

GENERALIZED LEGENDRE POLYNOMIALS AND RELATED CONGRUENCES MODULO p^2

ZHI-HONG SUN

School of Mathematical Sciences, Huaiyin Normal University,
Huaian, Jiangsu 223001, PR China
Email: zhihongsun@yahoo.com
Homepage: <http://www.hytc.edu.cn/xsjl/szh>

ABSTRACT. For any positive integer n and variables a and x we define the generalized Legendre polynomial $P_n(a, x) = \sum_{k=0}^n \binom{a}{k} \binom{-1-a}{k} \left(\frac{1-x}{2}\right)^k$. Let p be an odd prime. In the paper we prove many congruences modulo p^2 related to $P_{p-1}(a, x)$. For example, we show that $P_{p-1}(a, x) \equiv (-1)^{\langle a \rangle_p} P_{p-1}(a, -x) \pmod{p^2}$, where $\langle a \rangle_p$ is the least nonnegative residue of a modulo p . We also generalize some congruences of Zhi-Wei Sun, and determine $\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} 54^{-k}$ and $\sum_{k=0}^{p-1} \binom{a}{k} \binom{b-a}{k} \pmod{p^2}$, where $[x]$ is the greatest integer function. Finally we pose some supercongruences modulo p^2 concerning binary quadratic forms.

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1. Introduction.

For any positive integer n and variables a and x we introduce the generalized Legendre polynomial

$$(1.1) \quad \begin{aligned} P_n(a, x) &= \sum_{k=0}^n \binom{a}{k} \binom{-1-a}{k} \left(\frac{1-x}{2}\right)^k = \sum_{k=0}^n \binom{a}{k} \binom{a+k}{k} \left(\frac{x-1}{2}\right)^k \\ &= \sum_{k=0}^n \binom{a+k}{2k} \binom{2k}{k} \left(\frac{x-1}{2}\right)^k. \end{aligned}$$

We note that $\binom{-1-a}{k} = (-1)^k \binom{a+k}{k}$ and $\binom{a}{k} \binom{a+k}{k} = \binom{a+k}{2k} \binom{2k}{k}$. Clearly $P_n(a, x) = P_n(-1-a, x)$ and $P_n(n, x) = P_n(x)$ (see [B]), where $P_n(x)$ is the Legendre polynomial given by

$$(1.2) \quad P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (-1)^k \binom{2n-2k}{n} x^{n-2k} = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2 - 1)^n.$$

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Let $p > 3$ be a prime. In 2003, Rodriguez-Villegas[RV] conjectured the following congruences:

$$(1.3) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

$$(1.4) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \left(\frac{-3}{p}\right) \pmod{p^2},$$

$$(1.5) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2},$$

$$(1.6) \quad \sum_{k=0}^{p-1} \frac{\binom{3k}{k}\binom{6k}{3k}}{432^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol. These congruences were later confirmed by Mortenson[M1-M3] via the Gross-Koblitz formula. Recently Zhi-Wei Sun[Sun] posed more conjectures concerning the following sums modulo p^2 :

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} x^k, \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} x^k, \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k} x^k, \quad \sum_{k=0}^{p-1} \frac{\binom{3k}{k}\binom{6k}{3k}}{432^k} x^k.$$

As observed by Tauraso[T] and Zudilin, we have

$$\begin{aligned} \left(\frac{-\frac{1}{2}}{k}\right)^2 &= \frac{\binom{2k}{k}^2}{16^k}, \quad \left(\frac{-\frac{1}{3}}{k}\right)\left(\frac{-\frac{2}{3}}{k}\right) = \frac{\binom{2k}{k}\binom{3k}{k}}{27^k}, \\ \left(\frac{-\frac{1}{4}}{k}\right)\left(\frac{-\frac{3}{4}}{k}\right) &= \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}, \quad \left(\frac{-\frac{1}{6}}{k}\right)\left(\frac{-\frac{5}{6}}{k}\right) = \frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}. \end{aligned}$$

This is the motivation that we introduce and study $P_{p-1}(a, x) \pmod{p^2}$.

Let \mathbb{Z} be the set of integers. For a prime p let \mathbb{Z}_p denote the set of rational p -integers. For a p -integr a let $\langle a \rangle_p$ be the least nonnegative residue of a modulo p . Let p be an odd prime p and $a \in \mathbb{Z}_p$. In the paper we show that

$$(1.7) \quad P_{p-1}(a, x) \equiv (-1)^{\langle a \rangle_p} P_{p-1}(a, -x) \pmod{p^2}.$$

Taking $x = -1$ we obtain

$$(1.8) \quad \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} = P_{p-1}(a, -1) \equiv (-1)^{\langle a \rangle_p} \pmod{p^2}.$$

For $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ we get (1.3)-(1.6) immediately from (1.8). If $\langle a \rangle_p$ is odd, by (1.7) we have

$$(1.9) \quad \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \frac{1}{2^k} = P_{p-1}(a, 0) \equiv 0 \pmod{p^2}.$$

This generalizes previous special results in [S2] and [Su2]. If $f(0), f(1), \dots, f(p-1)$ are p -integers, we also prove the following general congruence:

$$(1.10) \quad \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \left((-1)^{\langle a \rangle_p} f(k) - \sum_{m=0}^k \binom{k}{m} (-1)^m f(m) \right) \equiv 0 \pmod{p^2}.$$

(1.7)-(1.9) can be viewed as vast generalizations of some congruences proved in [S2, S3] (with $a = -\frac{1}{2}$) and [Su2, Su3] (with $a = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$). When $\langle a \rangle_p \equiv 1 \pmod{2}$, taking $f(k) = \binom{2k}{k}/2^{2k}$ in (1.10) we get

$$(1.11) \quad \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \binom{2k}{k} \frac{1}{4^k} \equiv 0 \pmod{p^2}.$$

Inspired by the work of Zhi-Wei Sun ([Su2-Su4]) we establish the general congruence $(a+1)P_n(a+1, x) - (2a+1)xP_n(a, x) + aP_n(a-1, x) \equiv 0 \pmod{p^2}$ for $a \not\equiv 0, -1 \pmod{p}$.

and use it to prove our main results. As an application, we deduce the congruence for $\sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} 54^{-k} \pmod{p^2}$, see Theorem 3.4. In Section 4, we establish a general congruence for $\sum_{k=0}^{p-1} \binom{a}{k} \binom{b-a}{k} \pmod{p^2}$ and pose some conjectures for $\sum_{k=0}^{p-1} \binom{a}{k} \binom{b-a}{k} m^k \pmod{p^2}$, where p is an odd prime and $a, b \in \mathbb{Z}_p$.

2. General congruences for $P_{p-1}(a, x) \pmod{p^2}$.

Lemma 2.1. *Let n be a positive integer. Then*

$$\begin{aligned} & (a+1)P_n(a+1, x) - (2a+1)xP_n(a, x) + aP_n(a-1, x) \\ &= -2(2a+1) \binom{a}{n} \binom{a+n}{n} \left(\frac{x-1}{2} \right)^{n+1}. \end{aligned}$$

Proof. It is clear that

$$a(a+1-k)(a-k) + (a+1)(a+1+k)(a+k) = (2a+1)((a+k)(a-k+1) + 2k^2).$$

Thus,

$$\begin{aligned} & a \binom{a-1+k}{2k} + (a+1) \binom{a+1+k}{2k} - (2a+1) \binom{a+k}{2k} - (2a+1) \frac{k}{2k-1} \binom{a+k-1}{2k-2} \\ &= \frac{(a+k-1)(a+k-2) \cdots (a-(k-2))}{(2k)!} (a(a+1-k)(a-k) \\ &\quad + (a+1)(a+1+k)(a+k) - (2a+1)((a+k)(a-k+1) - 2k^2)) \\ &= 0. \end{aligned}$$

Therefore,

$$\begin{aligned}
 (2.1) \quad & a \binom{a-1+k}{2k} \binom{2k}{k} + (a+1) \binom{a+1+k}{2k} \binom{2k}{k} - (2a+1) \binom{a+k}{2k} \binom{2k}{k} \\
 & = (2a+1) \cdot 2 \binom{a+k-1}{2k-2} \binom{2k-2}{k-1}.
 \end{aligned}$$

Using (2.1) we deduce that

$$\begin{aligned}
 & (a+1)P_n(a+1, x) - (2a+1)xP_n(a, x) + aP_n(a-1, x) \\
 & = \sum_{k=0}^n \left\{ (a+1) \binom{a+1+k}{2k} \binom{2k}{k} + a \binom{a-1+k}{2k} \binom{2k}{k} \right. \\
 & \quad \left. - (2a+1) \left(1 + 2 \cdot \frac{x-1}{2} \right) \binom{a+k}{2k} \binom{2k}{k} \right\} \left(\frac{x-1}{2} \right)^k \\
 & = -(2a+1) \cdot 2 \binom{a+n}{2n} \binom{2n}{n} \left(\frac{x-1}{2} \right)^{n+1} \\
 & \quad + \sum_{k=0}^n \left\{ (a+1) \binom{a+1+k}{2k} \binom{2k}{k} + a \binom{a-1+k}{2k} \binom{2k}{k} \right. \\
 & \quad \left. - (2a+1) \left(\binom{a+k}{2k} \binom{2k}{k} + 2 \binom{a+k-1}{2k-2} \binom{2k-2}{k-1} \right) \right\} \left(\frac{x-1}{2} \right)^k \\
 & = -2(2a+1) \binom{a}{n} \binom{a+n}{n} \left(\frac{x-1}{2} \right)^{n+1}.
 \end{aligned}$$

This proves the lemma.

Theorem 2.1. *Let p be an odd prime and $a \in \mathbb{Z}_p$. Then*

$$\begin{aligned}
 & (a+1)P_{p-1}(a+1, x) - (2a+1)xP_{p-1}(a, x) + aP_{p-1}(a-1, x) \\
 & \equiv \begin{cases} -2p^2 \left(\frac{1}{\langle a \rangle_p} + \frac{1}{\langle a \rangle_p + 1} \right) \frac{a - \langle a \rangle_p}{p} \left(1 + \frac{a - \langle a \rangle_p}{p} \right) \left(\frac{x-1}{2} \right)^p \pmod{p^3} & \text{if } a \not\equiv 0, -1 \pmod{p}, \\ -2a(p+a+1) \left(\frac{x-1}{2} \right)^p \pmod{p^3} & \text{if } a \equiv 0 \pmod{p}, \\ 2(a+1)(a-p) \left(\frac{x-1}{2} \right)^p \pmod{p^3} & \text{if } a \equiv -1 \pmod{p}. \end{cases}
 \end{aligned}$$

Proof. Clearly

$$\binom{a}{p-1} \binom{a+p-1}{p-1} = \frac{a(a-1) \cdots (a-(p-2))(a+1)(a+2) \cdots (a+p-1)}{(p-1)!^2}.$$

If $\langle a \rangle_p \neq 0, p-1$, then

$$\begin{aligned}
 & a(a-1) \cdots (a-(p-2))(a+1)(a+2) \cdots (a+p-1) \\
 & \equiv (a - \langle a \rangle_p)(a + p - \langle a \rangle_p) \{ \langle a \rangle_p (\langle a \rangle_p - 1) \cdots 2 \cdot 1 \cdot (p-1)(p-2) \cdots (\langle a \rangle_p + 2) \\
 & \quad \times (\langle a \rangle_p + 1)(\langle a \rangle_p + 2) \cdots (p-1) \cdot 1 \cdot 2 \cdots (\langle a \rangle_p - 1) \} \\
 & = p^2 \cdot \frac{a - \langle a \rangle_p}{p} \left(1 + \frac{a - \langle a \rangle_p}{p} \right) \frac{(p-1)!^2}{\langle a \rangle_p (\langle a \rangle_p + 1)} \pmod{p^3}.
 \end{aligned}$$

If $\langle a \rangle_p = 0$, then

$$\begin{aligned}
\binom{a}{p-1} \binom{a+p-1}{p-1} &= a(a+p-1) \frac{(a^2-1^2)(a^2-2^2) \cdots (a^2-(p-2)^2)}{(p-1)!^2} \\
&\equiv a(a+p-1) \frac{(-1^2)(-2^2) \cdots (-(p-2)^2)}{(p-1)!^2} \equiv -\frac{a(a+p-1)}{(p-1)^2} \\
&\equiv -a(a+p-1)(p+1)^2 \equiv -a(a+p-1)(2p+1) \\
&\equiv -a(-2p+a+p-1) = a(p+1-a) \pmod{p^3}.
\end{aligned}$$

If $\langle a \rangle_p = p-1$, then $-1-a \equiv 0 \pmod{p}$ and so

$$\begin{aligned}
\binom{a}{p-1} \binom{a+p-1}{p-1} &= \binom{-1-a}{p-1} \binom{-1-a+p-1}{p-1} \\
&\equiv (-1-a)(p+1-(-1-a)) = -(a+1)(p+a+2) \pmod{p^3}.
\end{aligned}$$

Now putting all the above together with Lemma 2.1 in the case $n = p-1$ we deduce the result.

Corollary 2.1. *Let p be an odd prime and $a \in \mathbb{Z}_p$ with $a \not\equiv 0, -1 \pmod{p}$. Let $P'_n(a, x) = \frac{d}{dx} P_n(a, x)$. Then*

$$\begin{aligned}
&(a+1)P'_{p-1}(a+1, x) - (2a+1)xP'_{p-1}(a, x) - (2a+1)P_{p-1}(a, x) + aP'_{p-1}(a-1, x) \\
&\equiv 0 \pmod{p^2}.
\end{aligned}$$

Proof. By Theorem 2.1 we have

$$(a+1)P_{p-1}(a+1, x) - (2a+1)xP_{p-1}(a, x) + aP_{p-1}(a-1, x) = p^2 f(x),$$

where $f(x)$ is a polynomial of degree at most $p-1$ with rational p -integral coefficients. Taking derivatives on both sides we deduce the result.

Lemma 2.2. *Let p be an odd prime and let t and x be p -adic integers. Then*

$$P_{p-1}(pt, x) \equiv 1 - t + t \left(\frac{1+x}{2} \right)^p + t \left(\frac{1-x}{2} \right)^p \pmod{p^2} \quad \text{and so} \quad P_{p-1}(pt, x) \equiv 1 \pmod{p}.$$

Proof. It is clear that

$$\begin{aligned}
P_{p-1}(pt, x) &= \sum_{k=0}^{p-1} \binom{pt}{k} \binom{pt+k}{k} \left(\frac{x-1}{2} \right)^k \\
&= 1 + t \sum_{k=1}^{p-1} \frac{p}{k} \binom{pt-1}{k-1} \binom{pt+k}{k} \left(\frac{x-1}{2} \right)^k \\
&\equiv 1 + t \sum_{k=1}^{p-1} \binom{p}{k} \left(\frac{x-1}{2} \right)^k = 1 + t \left\{ \left(1 + \frac{x-1}{2} \right)^p - 1 - \left(\frac{x-1}{2} \right)^p \right\} \\
&= 1 - t + t \left\{ \left(\frac{1+x}{2} \right)^p + \left(\frac{1-x}{2} \right)^p \right\} \pmod{p^2}.
\end{aligned}$$

Since $\left(\frac{1 \pm x}{2} \right)^p \equiv \frac{1 \pm x^p}{2} \pmod{p}$, by the above we obtain $P_{p-1}(pt, x) \equiv 1 \pmod{p}$. This proves the lemma.

Lemma 2.3. *Let p be an odd prime and let t and x be p -adic integers. Then*

$$\begin{aligned} P_{p-1}(1+pt, x) &\equiv (1-t)x + pt \frac{x-1}{2} + t \left\{ (p+1) \frac{x-1}{2} \left(\left(\frac{1+x}{2} \right)^p + \left(\frac{1-x}{2} \right)^p \right) \right. \\ &\quad \left. + (p+1)^2 \left(\frac{1-x}{2} \right)^p + \left(\frac{1+x}{2} \right)^{p+1} - \left(\frac{1-x}{2} \right)^{p+1} \right\} \pmod{p^2} \end{aligned}$$

and so $P_{p-1}(1+pt, x) \equiv x \pmod{p}$.

Proof. It is clear that

$$\begin{aligned} P_{p-1}(1+pt, x) &= \sum_{k=0}^{p-1} \binom{pt+1}{k} \binom{pt+1+k}{k} \left(\frac{x-1}{2} \right)^k \\ &= 1 + (pt+1)(pt+2) \frac{x-1}{2} + \sum_{k=2}^{p-1} \frac{pt+1}{k} \cdot \frac{pt}{k-1} \binom{pt-1}{k-2} \binom{pt+1+k}{k} \left(\frac{x-1}{2} \right)^k \\ &\equiv 1 + (3pt+2) \frac{x-1}{2} + t \sum_{k=2}^{p-1} \binom{p+1}{k} \binom{k+1}{k} \left(\frac{x-1}{2} \right)^k \\ &= 1 + (pt+2-2t) \frac{x-1}{2} + t \sum_{k=1}^{p-1} \binom{p+1}{k} (k+1) \left(\frac{x-1}{2} \right)^k \pmod{p^2} \end{aligned}$$

and so

$$\begin{aligned} P_{p-1}(1+pt, x) &\equiv 1 + (pt+2-2t) \frac{x-1}{2} \\ &\quad + t \sum_{k=1}^{p-1} \binom{p+1}{k} \left(\frac{x-1}{2} \right)^k + t(p+1) \frac{x-1}{2} \sum_{k=1}^{p-1} \binom{p}{k-1} \left(\frac{x-1}{2} \right)^{k-1} \\ &= 1 + (pt+2-2t) \frac{x-1}{2} + t \left\{ \left(1 + \frac{x-1}{2} \right)^{p+1} - 1 - (p+1) \left(\frac{x-1}{2} \right)^p - \left(\frac{x-1}{2} \right)^{p+1} \right. \\ &\quad \left. + (p+1) \frac{x-1}{2} \left(\left(1 + \frac{x-1}{2} \right)^p - \left(\frac{x-1}{2} \right)^p - p \left(\frac{x-1}{2} \right)^{p-1} \right) \right\} \\ &= (1-t)x + pt \frac{x-1}{2} + t \left\{ (p+1) \frac{x-1}{2} \left(\left(\frac{1+x}{2} \right)^p + \left(\frac{1-x}{2} \right)^p \right) \right. \\ &\quad \left. + (p+1)^2 \left(\frac{1-x}{2} \right)^p + \left(\frac{1+x}{2} \right)^{p+1} - \left(\frac{1-x}{2} \right)^{p+1} \right\} \pmod{p^2}. \end{aligned}$$

Observe that $\left(\frac{1 \pm x}{2} \right)^p \equiv \frac{1 \pm x^p}{2} \pmod{p}$. From the above we see that

$$\begin{aligned} P_{p-1}(1+pt, x) &\equiv (1-t)x + t \left\{ \frac{x-1}{2} + \frac{1-x^p}{2} + \frac{1+x}{2} \cdot \frac{1+x^p}{2} - \frac{1-x}{2} \cdot \frac{1-x^p}{2} \right\} \\ &= x - tx + tx = x \pmod{p}. \end{aligned}$$

This completes the proof.

Theorem 2.2. *Let p be an odd prime and $a \in \mathbb{Z}_p$. Then*

$$P_{p-1}(a, x) \equiv (-1)^{\langle a \rangle_p} P_{p-1}(a, -x) \pmod{p^2}$$

and so

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} (x^k - (-1)^{\langle a \rangle_p} (1-x)^k) \equiv 0 \pmod{p^2}.$$

Proof. Suppose $m \in \{1, 2, \dots, p-2\}$ and $t \in \mathbb{Z}_p$. From Theorem 2.1 we have

$$\begin{aligned} (m+1+pt)P_{p-1}(m+1+pt, \pm x) - (2(m+pt)+1)(\pm x)P_{p-1}(m+pt, \pm x) \\ + (m+pt)P_{p-1}(m-1+pt, \pm x) \equiv 0 \pmod{p^2}. \end{aligned}$$

Thus,

$$\begin{aligned} (2.2) \quad & (m+1+pt)(P_{p-1}(m+1+pt, x) - (-1)^{m+1}P_{p-1}(m+1+pt, -x)) \\ & \equiv (2(m+pt)+1)x(P_{p-1}(m+pt, x) - (-1)^m P_{p-1}(m+pt, -x)) \\ & - (m+pt)(P_{p-1}(m-1+pt, x) - (-1)^{m-1}P_{p-1}(m-1+pt, -x)) \pmod{p^2}. \end{aligned}$$

From Lemma 2.2 we know that

$$P_{p-1}(pt, x) \equiv P_{p-1}(pt, -x) \pmod{p^2}.$$

From Lemma 2.3 we see that

$$\begin{aligned} & P_{p-1}(1+pt, x) + P_{p-1}(1+pt, -x) \\ & \equiv pt \left(\frac{x-1}{2} + \frac{-x-1}{2} \right) + t \left\{ (p+1) \left(\frac{x-1}{2} + \frac{-x-1}{2} \right) \left(\left(\frac{1+x}{2} \right)^p + \left(\frac{1-x}{2} \right)^p \right) \right. \\ & \quad \left. + (1+2p) \left(\left(\frac{1+x}{2} \right)^p + \left(\frac{1-x}{2} \right)^p \right) \right\} \\ & \equiv -pt + pt \left(\left(\frac{1+x}{2} \right)^p + \left(\frac{1-x}{2} \right)^p \right) \equiv -pt + pt \left(\frac{1+x^p}{2} + \frac{1-x^p}{2} \right) \\ & = 0 \pmod{p^2}. \end{aligned}$$

Thus,

$$P_{p-1}(m+pt, x) - (-1)^m P_{p-1}(m+pt, -x) \equiv 0 \pmod{p^2} \quad \text{for } m = 0, 1.$$

By (2.2) and induction we deduce that $P_{p-1}(m+pt, x) - (-1)^m P_{p-1}(m+pt, -x) \equiv 0 \pmod{p^2}$ for all $m = 0, 1, \dots, p-1$. Thus, $P_{p-1}(a, x) \equiv (-1)^{\langle a \rangle_p} P_{p-1}(a, -x)$ and hence

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} x^k &= P_{p-1}(a, 1-2x) \equiv (-1)^{\langle a \rangle_p} P_{p-1}(a, 2x-1) \\ &= (-1)^{\langle a \rangle_p} \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} (1-x)^k \pmod{p^2}. \end{aligned}$$

This completes the proof.

Remark 2.1 In the case $a = -\frac{1}{2}$, Theorem 2.2 was given by the author in [S2]. In the cases $a = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$, Theorem 2.2 was given by Z. W. Sun in [Su3].

Corollary 2.2. *Let p be an odd prime and $a \in \mathbb{Z}_p$. Then*

$$P_{p-1}(a, -1) = \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \equiv (-1)^{\langle a \rangle_p} \pmod{p^2}.$$

Proof. Taking $x = 1$ in Theorem 2.2 we obtain the result.

As mentioned in Section 1, taking $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ in Corollary 2.2 we deduce (1.3)-(1.6).

Corollary 2.3. *Let p be an odd prime and $a \in \mathbb{Z}_p$ with $\langle a \rangle_p \equiv 1 \pmod{2}$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \frac{1}{2^k} \equiv 0 \pmod{p^2}.$$

Proof. Taking $x = \frac{1}{2}$ in Theorem 2.2 we obtain the result.

Putting $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ in Corollary 2.3 we deduce the following congruences

$$(2.3) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{32^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 3 \pmod{4},$$

$$(2.4) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{54^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 2 \pmod{3},$$

$$(2.5) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 5, 7 \pmod{8},$$

$$(2.6) \quad \sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{6k}{3k}}{864^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 3 \pmod{4},$$

where p is a prime greater than 3. We remark that (2.3) was conjectured by Z. W. Sun and proved by the author in [S2], and (2.4) was conjectured by the author in [S2] and proved by Z. W. Sun in [Su4]. (2.5) and (2.6) were conjectured by Z. W. Sun and finally proved by him in [Su4], although the author proved the congruences modulo p earlier.

Lemma 2.4. *Let n be a positive integer. Then*

$$\sum_{k=0}^n \frac{\binom{a}{k} \binom{-1-a}{k}}{k+1} = \frac{\binom{a-1}{n} \binom{-2-a}{n}}{n+1}.$$

Proof. Set

$$f(n) = \frac{\binom{a}{n} \binom{-1-a}{n}}{n+1} \quad \text{and} \quad g(n) = \frac{\binom{a-1}{n} \binom{-2-a}{n}}{n+1}.$$

It is easily seen that $g(n) - g(n-1) = f(n)$. Thus,

$$\sum_{k=0}^n f(k) = f(0) + \sum_{k=1}^n (g(k) - g(k-1)) = f(0) - g(0) + g(n) = g(n).$$

This proves the lemma.

Theorem 2.3. *Let p be an odd prime, $a \in \mathbb{Z}_p$ and let m be a positive integer with $m < p$. Then*

$$\sum_{k=m}^{p-1} \binom{a}{k} \binom{-1-a}{k} \binom{k}{m} (x^{k-m} - (-1)^{m+\langle a \rangle_p} (1-x)^{k-m}) \equiv 0 \pmod{p^2}$$

and

$$\begin{aligned} & \sum_{k=0}^{p-2} \binom{a}{k} \binom{-1-a}{k} \frac{x^{k+1} + (-1)^{\langle a \rangle_p} (1-x)^{k+1}}{k+1} \\ & \equiv \begin{cases} 0 \pmod{p^2} & \text{if } a \not\equiv 0, -1 \pmod{p}, \\ 1 - 2a - a(x^p - (x-1)^p - 1)/p \pmod{p^2} & \text{if } a \equiv 0 \pmod{p}, \\ 2a + 3 + (1+a)(x^p - (x-1)^p - 1)/p \pmod{p^2} & \text{if } a \equiv -1 \pmod{p}. \end{cases} \end{aligned}$$

Proof. By Theorem 2.2,

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} (x^k - (-1)^{\langle a \rangle_p} (1-x)^k) = p^2 f(x),$$

where $f(x)$ is a polynomial of x with rational p -integral coefficients and degree at most $p-1$. Since $\frac{d^m x^k}{dx^m} = m! \binom{k}{m} x^{k-m}$ and $\frac{d^m (1-x)^k}{dx^m} = (-1)^k m! \binom{k}{m} (x-1)^{k-m} = (-1)^m m! \binom{k}{m} (1-x)^{k-m}$ for $k \geq m$, we see that

$$\sum_{k=m}^{p-1} \binom{a}{k} \binom{-1-a}{k} \binom{k}{m} (x^{k-m} - (-1)^{m+\langle a \rangle_p} (1-x)^{k-m}) = \frac{p^2}{m!} \cdot \frac{d^m}{dx^m} f(x).$$

As $\frac{d^m}{dx^m}f(x)$ is a polynomial of x with rational p -integral coefficients and degree at most $p-1$, we deduce the first result.

Set

$$\delta_p(a) = \begin{cases} 0 & \text{if } a \not\equiv 0, -1 \pmod{p}, \\ a & \text{if } a \equiv 0 \pmod{p}, \\ -1 - a & \text{if } a \equiv -1 \pmod{p}. \end{cases}$$

It is easy to see that

$$\binom{a}{p-1} \binom{-1-a}{p-1} \equiv \delta_p(a) \pmod{p^2}.$$

Thus, by Theorem 2.2 we have

$$\sum_{k=0}^{p-2} \binom{a}{k} \binom{-1-a}{k} (x^k - (-1)^{\langle a \rangle_p} (1-x)^k) + \delta_p(a) (x^{p-1} - (1-x)^{p-1}) = p^2 g(x),$$

where $g(x)$ is a polynomial of x with rational p -integral coefficients and degree at most $p-1$. It is easy to see that

$$\begin{aligned} p^2 \int_0^x g(t) dt \\ = \sum_{k=0}^{p-2} \binom{a}{k} \binom{-1-a}{k} \frac{x^{k+1} + (-1)^{\langle a \rangle_p} ((1-x)^{k+1} - 1)}{k+1} + \delta_p(a) \frac{x^p - (x-1)^p - 1}{p}. \end{aligned}$$

Thus, using Lemma 2.4 we get

$$\begin{aligned} & \sum_{k=0}^{p-2} \binom{a}{k} \binom{-1-a}{k} \frac{x^{k+1} + (-1)^{\langle a \rangle_p} (1-x)^{k+1}}{k+1} \\ & \equiv \sum_{k=0}^{p-2} \frac{\binom{a}{k} \binom{-1-a}{k}}{k+1} - \delta_p(a) \frac{x^p - (x-1)^p - 1}{p} \\ & = \frac{1}{p-1} \binom{a-1}{p-2} \binom{-2-a}{p-2} - \delta_p(a) \frac{x^p - (x-1)^p - 1}{p} \pmod{p^2}. \end{aligned}$$

If $a \not\equiv 0, -1 \pmod{p}$, then $\delta_p(a) = 0$ and

$$\binom{a-1}{p-2} \binom{-2-a}{p-2} = \frac{a}{p-1} \cdot \frac{p-1}{-1-a} \binom{a}{p-1} \binom{-1-a}{p-1} \equiv 0 \pmod{p^2}.$$

If $a \equiv 0 \pmod{p}$, then $\delta_p(a) = a$ and

$$\begin{aligned} \frac{\binom{a-1}{p-2} \binom{-2-a}{p-2}}{p-1} &= \frac{(a-1)(a-2) \cdots (a-(p-2))(-2-a) \cdots (-(p-1)-a)}{(p-2)!(p-1)!} \\ &\equiv \frac{(p-2)!(1-a \sum_{i=1}^{p-2} \frac{1}{i}) \cdot (p-1)!(1+a \sum_{i=2}^{p-1} \frac{1}{i})}{(p-2)!(p-1)!} \\ &\equiv (1+a/(p-1))(1-a) \equiv 1-2a \pmod{p^2}. \end{aligned}$$

If $a \equiv -1 \pmod{p}$, then $\delta_p(a) = -1 - a$. Using the above we get

$$\begin{aligned} \frac{1}{p-1} \binom{a-1}{p-2} \binom{-2-a}{p-2} &= \frac{1}{p-1} \binom{-1-a-1}{p-2} \binom{-2-(-1-a)}{p-2} \\ &\equiv 1 - 2(-1-a) = 2a + 3 \pmod{p^2}. \end{aligned}$$

Now putting all the above together we deduce the result.

Corollary 2.4. *Let p be an odd prime, $a \in \mathbb{Z}_p$ and let m be a positive integer with $m < p$. Then*

$$\sum_{k=m}^{p-1} \binom{a}{k} \binom{-1-a}{k} \binom{k}{m} \equiv (-1)^{m+\langle a \rangle_p} \binom{a}{m} \binom{-1-a}{m} \pmod{p^2}.$$

Proof. Taking $x = 1$ in Theorem 2.3 we obtain the result.

Theorem 2.4. *Let p be an odd prime and $a \in \mathbb{Z}_p$. If $f(0), f(1), \dots, f(p-1)$ are p -integers, then*

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \left((-1)^{\langle a \rangle_p} f(k) - \sum_{m=0}^k \binom{k}{m} (-1)^m f(m) \right) \equiv 0 \pmod{p^2}.$$

Proof. Using Corollary 2.4 we see that

$$\begin{aligned} &\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \sum_{m=0}^k \binom{k}{m} (-1)^m f(m) \\ &= \sum_{m=0}^{p-1} (-1)^m f(m) \sum_{k=m}^{p-1} \binom{a}{k} \binom{-1-a}{k} \binom{k}{m} \\ &\equiv \sum_{m=0}^{p-1} (-1)^m f(m) \cdot (-1)^{m+\langle a \rangle_p} \binom{a}{m} \binom{-1-a}{m} \\ &= (-1)^{\langle a \rangle_p} \sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} f(k) \pmod{p^2}. \end{aligned}$$

This yields the result.

Remark 2.2 In the case $a = -\frac{1}{2}$, Corollary 2.3 and Theorem 2.4 were obtained by the author in [S3]. In the cases $a = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$, Corollary 2.3 and Theorem 2.4 were recently obtained by Z.W. Sun in [Su3].

Theorem 2.5. *Let p be an odd prime and $a \in \mathbb{Z}_p$ with $\langle a \rangle_p \equiv 1 \pmod{2}$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{-1-a}{k} \binom{2k}{k} \frac{1}{2^{2k}} \equiv 0 \pmod{p^2}.$$

Proof. Set $f(k) = \binom{2k}{k}/2^{2k}$. From [S1, Example 10] we know that $\sum_{m=0}^k \binom{k}{m} (-1)^m f(m) = f(k)$. Thus, applying Theorem 2.4 we deduce the result.

Putting $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ in Theorem 2.5 we deduce that for any prime $p > 3$,

$$(2.7) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 3 \pmod{4},$$

$$(2.8) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 5 \pmod{6},$$

$$(2.9) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{256^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 5, 7 \pmod{8},$$

$$(2.10) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{432^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 3 \pmod{4}.$$

Here (2.7) was conjectured by Beukers[Be] in 1985 and proved by van Hamme[vH]. (2.8)-(2.10) were conjectured by Rodriguez-Villegas[RV] and proved by Z. W. Sun[Su4].

3. Congruences for $P_{p-1}(a, 0) \pmod{p^2}$.

For given positive integer n and prime p we define

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad \text{and} \quad q_p(a) = \frac{a^{p-1} - 1}{p}.$$

Theorem 3.1. *Let p be an odd prime and $t \in \mathbb{Z}_p$.*

- (i) *If $n \in \{0, 1, \dots, \frac{p-3}{2}\}$, then $P_{p-1}(2n+1+pt, 0) \equiv 0 \pmod{p^2}$.*
- (ii) *If $n \in \{0, 1, \dots, \frac{p-1}{2}\}$, then*

$$P_{p-1}(2n+pt, 0) \equiv \binom{\frac{p-1}{2}}{n} \left(1 + p \left((1+t)H_{2n} - \frac{2t+1}{2}H_n - tq_p(2) \right) \right) \pmod{p^2}.$$

Proof. Putting $x = 0$ in Theorem 2.1 we see that

$$P_{p-1}(a+1, 0) \equiv -\frac{a}{a+1} P_{p-1}(a-1, 0) \pmod{p^2} \quad \text{for } a \not\equiv 0, -1 \pmod{p}.$$

Assume $n \in \{1, 2, \dots, \frac{p-3}{2}\}$. Then

$$\begin{aligned} & P_{p-1}(2n+1+pt, 0) \\ &= -\frac{2n+pt}{2n+1+pt} P_{2n-1+pt}(0) = \dots \\ &= (-1)^n \frac{2n+pt}{2n+1+pt} \cdot \frac{2n-2+pt}{2n-1+pt} \dots \frac{2+pt}{3+pt} P_{p-1}(1+pt, 0) \pmod{p^2}. \end{aligned}$$

By Lemma 2.3 we have

$$P_{p-1}(1+pt, 0) \equiv -\frac{p}{2}t + t \left(-\frac{p+1}{2} \cdot \frac{1}{2^{p-1}} + (2p+1) \cdot \frac{1}{2^p} \right) = -\frac{p}{2}t + \frac{p}{2^p}t \equiv 0 \pmod{p^2}.$$

Thus, from the above we deduce that $P_{p-1}(2n+1+pt, 0) \equiv 0 \pmod{p^2}$ for $n = 0, 1, \dots, \frac{p-3}{2}$. This proves (i).

Now let us consider (ii). Assume $n \in \{1, 2, \dots, \frac{p-1}{2}\}$. Then

$$\begin{aligned} & P_{p-1}(2n+pt, 0) \\ &= -\frac{2n-1+pt}{2n+pt} P_{2n-2+pt}(0) = \dots \\ &= (-1)^n \frac{2n-1+pt}{2n+pt} \cdot \frac{2n-3+pt}{2n-2+pt} \dots \frac{3+pt}{4+pt} \cdot \frac{1+pt}{2+pt} P_{p-1}(pt, 0) \\ &\equiv (-1)^n \frac{1 \cdot 3 \dots (2n-1)(1+pt \sum_{k=1}^n \frac{1}{2k-1})}{2 \cdot 4 \dots (2n)(1+pt \sum_{k=1}^n \frac{1}{2k})} P_{p-1}(pt, 0) \\ &\equiv \frac{1}{(-4)^n} \binom{2n}{n} \left(1+pt \sum_{k=1}^n \frac{1}{2k-1} \right) \left(1-pt \sum_{k=1}^n \frac{1}{2k} \right) P_{p-1}(pt, 0) \\ &\equiv \frac{1}{(-4)^n} \binom{2n}{n} (1+pt(H_{2n} - H_n)) P_{p-1}(pt, 0) \pmod{p^2}. \end{aligned}$$

By [S1, Lemma 2.4] we have

$$(3.1) \quad \frac{1}{(-4)^n} \binom{2n}{n} \equiv \binom{\frac{p-1}{2}}{n} \left(1+p \sum_{k=1}^n \frac{1}{2k-1} \right) = \binom{\frac{p-1}{2}}{n} \left(1+p(H_{2n} - \frac{1}{2}H_n) \right) \pmod{p^2}.$$

From Lemma 2.2 we have

$$P_{p-1}(pt, 0) \equiv 1 - t + \frac{t}{2^{p-1}} \equiv 1 - t + t(1 - pq_p(2)) = 1 - ptq_p(2) \pmod{p^2}.$$

Therefore, for $n = 0, 1, \dots, \frac{p-1}{2}$ we have

$$\begin{aligned} P_{p-1}(2n+pt, 0) &\equiv \binom{\frac{p-1}{2}}{n} (1+p(H_{2n} - \frac{H_n}{2}))(1+pt(H_{2n} - H_n))(1-ptq_p(2)) \\ &\equiv \binom{\frac{p-1}{2}}{n} (1+p((1+t)H_{2n} - \frac{2t+1}{2}H_n))(1-ptq_p(2)) \\ &\equiv \binom{\frac{p-1}{2}}{n} (1+p((1+t)H_{2n} - \frac{2t+1}{2}H_n - tq_p(2))) \pmod{p^2}. \end{aligned}$$

This proves (ii) and hence the proof is complete.

Corollary 3.1. *Let p be an odd prime and let a be a p -adic integer with $a \not\equiv 0 \pmod{p}$. Then $P_{p-1}(a, 0) \equiv 0 \pmod{p^2}$ or $P_{p-1}(-a, 0) \equiv 0 \pmod{p^2}$.*

Proof. If $a \equiv 2n \pmod{p}$ for some $n \in \{1, 2, \dots, \frac{p-1}{2}\}$, then $-a \equiv p - 2n = 1 + 2(\frac{p-1}{2} - n) \pmod{p}$. Thus the result follows from Theorem 3.1(i).

Theorem 3.2. *Let p be an odd prime, $r, m \in \mathbb{Z}$, $m \geq 1$, $r \in \{\pm 1, \pm 2, \dots, \pm m - 1\}$ and $(r, m) = 1$.*

(i) *If $2 \nmid rm$, then*

$$P_{p-1}\left(\frac{r}{m}, 0\right) = \sum_{k=0}^{p-1} \binom{r/m}{k} \binom{-1-r/m}{k} \frac{1}{2^k} \\ \equiv \begin{cases} \binom{(p-1)/2}{n} (1 + p((1 - \frac{2s}{m})H_{2n} + (\frac{2s}{m} - \frac{1}{2})H_n + \frac{2s}{m}q_p(2))) \pmod{p^2} \\ \text{if } p \equiv \frac{r}{2s} \pmod{m} \text{ for some } s \in \{1, 2, \dots, \frac{m-1}{2}\} \\ \text{with } (s, m) = 1 \text{ and } n = \frac{sp-(m+r)/2}{m}, \\ 0 \pmod{p^2} \quad \text{otherwise.} \end{cases}$$

(ii) *If $2 \nmid m$ and $2 \mid r$, then*

$$P_{p-1}\left(\frac{r}{m}, 0\right) = \sum_{k=0}^{p-1} \binom{r/m}{k} \binom{-1-r/m}{k} \frac{1}{2^k} \\ \equiv \begin{cases} \binom{(p-1)/2}{n} (1 + p((1 - \frac{2s}{m})H_{2n} + (\frac{2s}{m} - \frac{1}{2})H_n + \frac{2s}{m}q_p(2))) \pmod{p^2} \\ \text{if } p \equiv -\frac{r/2}{s} \pmod{m} \text{ for some } s \in \{1, 2, \dots, \frac{m-1}{2}\} \\ \text{with } (s, m) = 1 \text{ and } n = \frac{sp+r/2}{m}, \\ 0 \pmod{p^2} \quad \text{otherwise.} \end{cases}$$

(iii) *If $2 \mid m$ and $2 \nmid r$, then*

$$P_{p-1}\left(\frac{r}{m}, 0\right) = \sum_{k=0}^{p-1} \binom{r/m}{k} \binom{-1-r/m}{k} \frac{1}{2^k} \\ \equiv \begin{cases} \binom{(p-1)/2}{n} (1 + p((1 - \frac{s}{m})H_{2n} + (\frac{s}{m} - \frac{1}{2})H_n + \frac{s}{m}q_p(2))) \pmod{p^2} \\ \text{if } p \equiv -\frac{r}{s} \pmod{2m} \text{ for some } s \in \{1, 2, \dots, \frac{m}{2}\} \\ \text{with } (s, m) = 1 \text{ and } n = \frac{sp+r}{2m}, \\ 0 \pmod{p^2} \quad \text{otherwise.} \end{cases}$$

Proof. If $2 \nmid rm$, setting $n = \frac{sp-(m+r)/2}{m}$ and $t = -\frac{2s}{m}$ we find $2n + pt = -1 - \frac{r}{m}$. If $2 \nmid m$ and $2 \mid r$, setting $n = \frac{sp+r/2}{m}$ and $t = -\frac{2s}{m}$ we find $2n + pt = \frac{r}{m}$. If $2 \mid m$ and $2 \nmid r$, setting $n = \frac{sp+r}{2m}$ and $t = -\frac{s}{m}$ we find $2n + pt = \frac{r}{m}$. Now applying Theorem 3.1 and the fact $P_{p-1}(\frac{r}{m}, x) = P_{p-1}(-1 - \frac{r}{m}, x)$ we deduce the result.

From Theorem 3.2 we deduce the following result.

Theorem 3.3. *Let p be an odd prime. Then*

$$\begin{aligned}
P_{p-1}(1/2, 0) &\equiv 0 \pmod{p^2} \quad \text{for } p \equiv 1 \pmod{4}, \\
P_{p-1}(1/3, 0) &\equiv 0 \pmod{p^2} \quad \text{for } p \equiv 1 \pmod{3}, \\
P_{p-1}(1/4, 0) &\equiv 0 \pmod{p^2} \quad \text{for } p \equiv 1, 3 \pmod{8}, \\
P_{p-1}(1/5, 0) &\equiv 0 \pmod{p^2} \quad \text{for } p \equiv 1, 2 \pmod{5}, \\
P_{p-1}(1/6, 0) &\equiv 0 \pmod{p^2} \quad \text{for } p \equiv 1 \pmod{4}, \\
P_{p-1}(1/7, 0) &\equiv 0 \pmod{p^2} \quad \text{for } p \equiv 1, 3, 5 \pmod{7}, \\
P_{p-1}(1/8, 0) &\equiv 0 \pmod{p^2} \quad \text{for } p \equiv 1, 7, 11, 13 \pmod{16}, \\
P_{p-1}(1/9, 0) &\equiv 0 \pmod{p^2} \quad \text{for } p \equiv 1, 2, 4 \pmod{9}, \\
P_{p-1}(1/10, 0) &\equiv 0 \pmod{p^2} \quad \text{for } p \equiv 1, 3, 7, 9 \pmod{20}, \\
P_{p-1}(1/11, 0) &\equiv 0 \pmod{p^2} \quad \text{for } p \equiv 1, 4, 5, 8, 9 \pmod{11}, \\
P_{p-1}(1/12, 0) &\equiv 0 \pmod{p^2} \quad \text{for } p \equiv 1, 5, 7, 11 \pmod{24}.
\end{aligned}$$

Lemma 3.1. *Let p be an odd prime. Then*

- (i) ([L]) $H_{\frac{p-1}{2}} \equiv -2q_p(2) \pmod{p}$, $H_{[\frac{p}{4}]} \equiv -3q_p(2) \pmod{p}$.
- (ii) ([L]) For $p > 3$ we have $H_{[\frac{p}{3}]} \equiv -\frac{3}{2}q_p(3) \pmod{p}$ and $H_{[\frac{p}{6}]} \equiv -2q_p(2) - \frac{3}{2}q_p(3) \pmod{p}$.

Theorem 3.4. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{54^k} \equiv \begin{cases} 2A - \frac{p}{2A} \pmod{p^2} & \text{if } 3 \mid p-1, p = A^2 + 3B^2 \text{ and } 3 \mid A-1, \\ 0 \pmod{p^2} & \text{if } 3 \mid p-2. \end{cases}$$

Proof. From Theorem 3.2(i) we see that

$$\begin{aligned}
\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{54^k} &= P_{p-1}\left(-\frac{1}{3}, 0\right) \\
&\equiv \begin{cases} \left(\frac{\frac{p-1}{2}}{\frac{p-1}{3}}\right) \left(1 + p\left(\frac{1}{3}H_{\frac{2(p-1)}{3}} + \frac{1}{6}H_{\frac{p-1}{3}} + \frac{2}{3}q_p(2)\right)\right) \pmod{p^2} & \text{if } p \equiv 1 \pmod{3}, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}
\end{aligned}$$

Now we assume $p \equiv 1 \pmod{3}$ and $p = A^2 + 3B^2$ with $A, B \in \mathbb{Z}$ and $A \equiv 1 \pmod{3}$. From Lemma 3.1 we have $H_{\frac{p-1}{3}} \equiv -\frac{3}{2}q_p(3) \pmod{p}$ and

$$H_{\frac{2(p-1)}{3}} = H_{p-1} - \sum_{k=1}^{(p-1)/3} \frac{1}{p-k} \equiv \sum_{k=1}^{(p-1)/3} \frac{1}{k} \equiv -\frac{3}{2}q_p(3) \pmod{p}.$$

Thus,

$$\begin{aligned}\frac{1}{3}H_{\frac{2(p-1)}{3}} + \frac{1}{6}H_{\frac{p-1}{3}} + \frac{2}{3}q_p(2) &\equiv -\frac{1}{2}q_p(3) - \frac{1}{4}q_p(3) + \frac{2}{3}q_p(2) \\ &= \frac{2}{3}q_p(2) - \frac{3}{4}q_p(3) \pmod{p}.\end{aligned}$$

By [BEW, Theorem 9.4.4],

$$\binom{\frac{p-1}{2}}{\frac{p-1}{3}} = \binom{\frac{p-1}{2}}{\frac{p-1}{6}} \equiv \left(2A - \frac{p}{2A}\right) \left(1 - p\frac{2}{3}q_p(2) + p\frac{3}{4}q_p(3)\right) \pmod{p^2}.$$

Therefore,

$$\begin{aligned}P_{p-1}\left(-\frac{1}{3}, 0\right) &\equiv \binom{\frac{p-1}{2}}{\frac{p-1}{3}} \left(1 + p\left(\frac{1}{3}H_{\frac{2(p-1)}{3}} + \frac{1}{6}H_{\frac{p-1}{3}} + \frac{2}{3}q_p(2)\right)\right) \\ &\equiv \left(2A - \frac{p}{2A}\right) \left(1 - p\left(\frac{2}{3}q_p(2) - \frac{3}{4}q_p(3)\right)\right) \left(1 + p\left(\frac{2}{3}q_p(2) - \frac{3}{4}q_p(3)\right)\right) \\ &\equiv 2A - \frac{p}{2A} \pmod{p^2}.\end{aligned}$$

This completes the proof.

Remark 3.1 In [S1] the author conjectured Theorem 3.4 and proved the congruence modulo p . In [Su3], Zhi-Wei Sun proved the result for $p \equiv 2 \pmod{3}$.

4. Congruences for $\sum_{k=0}^{p-1} \binom{a}{k} \binom{b-a}{k} \pmod{p^2}$.

Let n be a nonnegative integer. For two variables a and b we define

$$(4.1) \quad S_n(a, b) = \sum_{k=0}^n \binom{a}{k} \binom{b-a}{k}.$$

Using Maple and Zeilberger's algorithm we find

$$(4.2) \quad (a-b)S_n(a, b) + (a+1)S_n(a+1, b) = (2a-b+1) \binom{a}{n} \binom{b-a-1}{n}.$$

Lemma 4.1. *Let p be an odd prime and $b, c, t \in \mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1} \binom{pt}{k} \binom{b+cpt}{k} \equiv 1 + pt \sum_{k=1}^{\langle b \rangle_p} \frac{1}{k} \pmod{p^2}.$$

Proof. Clearly

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{pt}{k} \binom{b+cpt}{k} \\
&= 1 + \sum_{k=1}^{p-1} \frac{pt}{k} \binom{pt-1}{k-1} \binom{b+cpt}{k} \equiv 1 + \sum_{k=1}^{p-1} \frac{pt}{k} \binom{-1}{k-1} \binom{b}{k} \\
&\equiv 1 + \sum_{k=1}^{p-1} \frac{pt}{k} (-1)^{k-1} \binom{\langle b \rangle_p}{k} = 1 + pt \sum_{k=1}^{\langle b \rangle_p} \frac{(-1)^{k-1}}{k} \binom{\langle b \rangle_p}{k} \\
&= 1 + pt \sum_{k=1}^{\langle b \rangle_p} (-1)^{k-1} \binom{\langle b \rangle_p}{k} \int_0^1 x^{k-1} dx = 1 + pt \int_0^1 \sum_{k=1}^{\langle b \rangle_p} \binom{\langle b \rangle_p}{k} (-x)^{k-1} dx \\
&= 1 + pt \int_0^1 \frac{(1-x)^{\langle b \rangle_p} - 1}{-x} dx = 1 + pt \int_0^1 \frac{u^{\langle b \rangle_p} - 1}{u-1} du \\
&= 1 + pt \int_0^1 \sum_{k=0}^{\langle b \rangle_p - 1} u^k du = 1 + pt \sum_{k=0}^{\langle b \rangle_p - 1} \frac{1}{k+1} \pmod{p^2}.
\end{aligned}$$

This proves the lemma.

Lemma 4.2. *Let p be an odd prime, $m \in \{1, 2, \dots, p-1\}$ and $t \in \mathbb{Z}_p$. Then*

$$\binom{m+pt-1}{p-1} \equiv \frac{pt}{m} - \frac{p^2 t^2}{m^2} + \frac{p^2 t}{m} \sum_{k=1}^m \frac{1}{k} \pmod{p^3}.$$

Proof. For $m < \frac{p}{2}$ we see that

$$\begin{aligned}
& \binom{m+pt-1}{p-1} \\
&= \frac{(m-1+pt)(m-2+pt) \cdots (1+pt) \cdot pt(pt-1) \cdots (pt-(p-1-m))}{(p-1)!} \\
&= \frac{pt(p^2 t^2 - 1^2) \cdots (p^2 t^2 - (m-1)^2) (pt-m) \cdots (pt-(p-1-m))}{(p-1)!} \\
&\equiv pt \frac{(m-1)!(-1)(-2) \cdots (-(p-1-m))}{(p-1)!} \left(1 - pt \sum_{k=m}^{p-1-m} \frac{1}{k} \right) \\
&= pt \cdot \frac{(-1)^{p-1-m} \cdot (m-1)!}{(p-m) \cdots (p-1)} \left(1 - pt \left(H_{p-1-m} - H_m + \frac{1}{m} \right) \right) \pmod{p^3}.
\end{aligned}$$

For $m > \frac{p}{2}$ we also have

$$\begin{aligned}
& \binom{m+pt-1}{p-1} \\
&= \frac{(m-1+pt)(m-2+pt) \cdots (1+pt) \cdot pt(pt-1) \cdots (pt-(p-1-m))}{(p-1)!} \\
&= \frac{pt(p^2t^2-1^2) \cdots (p^2t^2-(p-1-m)^2)(pt+p-m) \cdots (pt+m-1)}{(p-1)!} \\
&\equiv pt \frac{(-1)^{p-1-m}(p-1-m)!(m-1)!}{(p-1)!} \left(1 + pt \sum_{k=p-m}^{m-1} \frac{1}{k}\right) \\
&= pt \cdot \frac{(-1)^{p-1-m} \cdot (m-1)!}{(p-m) \cdots (p-1)} \left(1 - pt(H_{p-1-m} - H_m + \frac{1}{m})\right) \pmod{p^3}.
\end{aligned}$$

Since $(p-m) \cdots (p-1) \equiv (-1)^m m! (1 - pH_m) \pmod{p^2}$, by the above we get

$$\begin{aligned}
\binom{m+pt-1}{p-1} &\equiv pt \cdot \frac{(-1)^m \cdot (m-1)!}{(-1)^m \cdot m! (1 - pH_m)} \left(1 - \frac{pt}{m} - pt(H_{p-1-m} - H_m)\right) \\
&\equiv \frac{pt}{m} (1 + pH_m) \left(1 - \frac{pt}{m} - pt(H_{p-1-m} - H_m)\right) \\
&\equiv \frac{pt}{m} - \frac{p^2t}{m} \left(\frac{t}{m} + t(H_{p-1-m} - H_m) - H_m\right) \pmod{p^3}.
\end{aligned}$$

To see the result, we note that

$$\begin{aligned}
H_{p-1-m} - H_m &= \sum_{r=1}^{p-1-m} \frac{1}{r} - \sum_{k=1}^m \frac{1}{k} = \sum_{k=m+1}^{p-1} \frac{1}{p-k} - \sum_{k=1}^m \frac{1}{k} \\
&\equiv - \sum_{k=m+1}^{p-1} \frac{1}{k} - \sum_{k=1}^m \frac{1}{k} = - \sum_{r=1}^{(p-1)/2} \left(\frac{1}{r} + \frac{1}{p-r}\right) \\
&\equiv 0 \pmod{p}.
\end{aligned}$$

Lemma 4.3. *Let p be an odd prime, $m \in \{1, 2, \dots, p-1\}$ and $b, t \in \mathbb{Z}_p$. Then*

$$\binom{b-(m+pt)}{p-1} \equiv \begin{cases} 1 \pmod{p^2} & \text{if } m \equiv b+1 \pmod{p}, \\ \frac{b-\langle b \rangle_p - pt}{m-1-b} \pmod{p^2} & \text{if } m \leq \langle b \rangle_p \pmod{p}, \\ \frac{b-\langle b \rangle_p - p(t+1)}{m-1-b} \pmod{p^2} & \text{if } m > \langle b \rangle_p + 1 \pmod{p}. \end{cases}$$

Proof. If $m \equiv b+1 \pmod{p}$, setting $b-m = rp + p-1$ we find $r \in \mathbb{Z}_p$ and so

$$\begin{aligned}
\binom{b-m-pt}{p-1} &= \frac{(p-1+p(r-t))(p-2+p(r-t)) \cdots (1+p(r-t))}{(p-1)!} \\
&\equiv 1 + p(r-t) \sum_{k=1}^{p-1} \frac{1}{k} = 1 + p(r-t) \sum_{k=1}^{\frac{p-1}{2}} \left(\frac{1}{k} + \frac{1}{p-k}\right) \equiv 1 \pmod{p^2}.
\end{aligned}$$

Now we assume $m \not\equiv b+1 \pmod{p}$ and $b-m = \langle b-m \rangle_p + pr$. Then $r \in \mathbb{Z}_p$ and

$$\begin{aligned}
& \binom{b-(m+pt)}{p-1} \\
&= \frac{(\langle b-m \rangle_p + p(r-t))(\langle b-m \rangle_p - 1 + p(r-t)) \cdots (\langle b-m \rangle_p - p + 2 + p(r-t))}{(p-1)!} \\
&\equiv \langle b-m \rangle_p! \cdot p(r-t) \cdot (p-1)(p-2) \cdots (2 + \langle b-m \rangle_p) \\
&= \frac{p(r-t) \cdot (p-1)!}{1 + \langle b-m \rangle_p} \equiv \frac{b-m - \langle b-m \rangle_p - pt}{m-1-b} \pmod{p^2}.
\end{aligned}$$

Since

$$\langle b-m \rangle_p = \begin{cases} \langle b \rangle_p - m & \text{if } m \leq \langle b \rangle_p, \\ p + \langle b \rangle_p - m & \text{if } m > \langle b \rangle_p + 1, \end{cases}$$

we see that

$$b-m - \langle b-m \rangle_p = \begin{cases} b-m - (\langle b \rangle_p - m) = b - \langle b \rangle_p & \text{if } m \leq \langle b \rangle_p, \\ b-m - (p + \langle b \rangle_p - m) = b - \langle b \rangle_p - p & \text{if } m > \langle b \rangle_p + 1. \end{cases}$$

Now combining all the above we obtain the result.

Theorem 4.1. *Let p be an odd prime and $a, b \in \mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{b-a}{k} \equiv \begin{cases} \frac{(-1)^{\langle a \rangle_p - \langle b \rangle_p - 1}}{(\langle a \rangle_p - \langle b \rangle_p) \binom{\langle a \rangle_p}{\langle b \rangle_p}} (b - \langle b \rangle_p) \pmod{p^2} & \text{if } \langle a \rangle_p > \langle b \rangle_p, \\ \binom{\langle b \rangle_p}{\langle a \rangle_p} (1 + (b - \langle b \rangle_p) H_{\langle b \rangle_p} - (a - \langle a \rangle_p) H_{\langle a \rangle_p} - (b - a - \langle b - a \rangle_p) H_{\langle b-a \rangle_p}) \pmod{p^2} & \text{if } \langle a \rangle_p \leq \langle b \rangle_p. \end{cases}$$

Proof. If $\langle a \rangle_p = 0$, the result follows from Lemma 4.1. From now on we assume $\langle a \rangle_p \geq 1$. Set $a = \langle a \rangle_p + pt$ and $b = \langle b \rangle_p + ps$. Then $s, t \in \mathbb{Z}_p$. For $m \in \{1, 2, \dots, p-1\}$, by (4.2) and Lemmas 4.2-4.3 we obtain

$$\begin{aligned}
& (m+pt)S_{p-1}(m+pt, b) + (m+pt-b-1)S_{p-1}(m+pt-1, b) \\
&= (2(m+pt)-1-b) \binom{m+pt-1}{p-1} \binom{b-(m+pt)}{p-1} \\
(4.3) \quad & \equiv \begin{cases} (2m-1-b) \frac{pt}{m} + p^2 t \left(\frac{t}{m} + H_m \right) \pmod{p^3} & \text{if } m = \langle b \rangle_p + 1, \\ (2m-1-b) \frac{pt}{m} \cdot \frac{b-\langle b \rangle_p-pt}{m-1-b} \pmod{p^3} & \text{if } m \leq \langle b \rangle_p, \\ (2m-1-b) \frac{pt}{m} \cdot \frac{b-\langle b \rangle_p-p(t+1)}{m-1-b} \pmod{p^3} & \text{if } m > \langle b \rangle_p + 1. \end{cases}
\end{aligned}$$

Hence, if $1 \leq \langle a \rangle_p \leq \langle b \rangle_p$, then

$$\begin{aligned}
& S_{p-1}(a, b) \\
&= S_{p-1}(\langle a \rangle_p + pt, b) \equiv -\frac{\langle a \rangle_p + pt - b - 1}{\langle a \rangle_p + pt} S_{p-1}(\langle a \rangle_p - 1 + pt, b) \\
&= \frac{1 + \langle b \rangle_p - \langle a \rangle_p + p(s - t)}{\langle a \rangle_p + pt} S_{p-1}(\langle a \rangle_p - 1 + pt, b) \\
&\equiv \frac{1 + \langle b \rangle_p - \langle a \rangle_p + p(s - t)}{\langle a \rangle_p + pt} \cdot \frac{1 + \langle b \rangle_p - (\langle a \rangle_p - 1) + p(s - t)}{\langle a \rangle_p - 1 + pt} S_{p-1}(\langle a \rangle_p - 2 + pt, b) \\
&\equiv \dots \equiv \prod_{k=1}^{\langle a \rangle_p} \frac{1 + \langle b \rangle_p - k + p(s - t)}{k + pt} \cdot S_{p-1}(pt, b) \pmod{p^2}.
\end{aligned}$$

Now applying Lemma 4.1 we see that for $1 \leq \langle a \rangle_p \leq \langle b \rangle_p$,

$$\begin{aligned}
& S_{p-1}(a, b) \\
&\equiv \frac{\langle b \rangle_p (\langle b \rangle_p - 1) \cdots (\langle b \rangle_p - \langle a \rangle_p + 1) (1 + p(s - t)(H_{\langle b \rangle_p} - H_{\langle b - a \rangle_p}))}{\langle a \rangle_p! (1 + pt H_{\langle a \rangle_p})} (1 + pt H_{\langle b \rangle_p}) \\
&\equiv \left(\frac{\langle b \rangle_p}{\langle a \rangle_p} \right) (1 + p(s - t)(H_{\langle b \rangle_p} - H_{\langle b - a \rangle_p})) (1 - pt H_{\langle a \rangle_p}) (1 + pt H_{\langle b \rangle_p}) \\
&\equiv \left(\frac{\langle b \rangle_p}{\langle a \rangle_p} \right) (1 + ps H_{\langle b \rangle_p} - pt H_{\langle a \rangle_p} - (ps - pt) H_{\langle b - a \rangle_p}) \\
&= \left(\frac{\langle b \rangle_p}{\langle a \rangle_p} \right) (1 + (b - \langle b \rangle_p) H_{\langle b \rangle_p} - (a - \langle a \rangle_p) H_{\langle a \rangle_p} - (b - a - \langle b - a \rangle_p) H_{\langle b - a \rangle_p}) \pmod{p^2}.
\end{aligned}$$

Now we assume $\langle a \rangle_p > \langle b \rangle_p$. Clearly

$$S_{p-1}(\langle b \rangle_p + pt, b) = 1 + \sum_{k=1}^{p-1} \binom{\langle b \rangle_p + pt}{k} \frac{b - \langle b \rangle_p - pt}{k} \binom{b - \langle b \rangle_p - pt - 1}{k-1} \equiv 1 \pmod{p}.$$

By (4.2) and (4.3) we have

$$\begin{aligned}
& (\langle b \rangle_p + 1 + pt) S_{p-1}(\langle b \rangle_p + 1 + pt, b) \\
&\equiv -(\langle b \rangle_p + pt - b) S_{p-1}(\langle b \rangle_p + pt, b) + (2m - 1 - b) \frac{pt}{m} \\
&\equiv b - \langle b \rangle_p - pt + pt = b - \langle b \rangle_p \pmod{p^2}.
\end{aligned}$$

Thus,

$$(4.4) \quad S_{p-1}(\langle b \rangle_p + 1 + pt, b) \equiv \frac{b - \langle b \rangle_p}{\langle b \rangle_p + 1 + pt} \equiv \frac{b - \langle b \rangle_p}{\langle b \rangle_p + 1} \pmod{p^2}.$$

Now from (4.2)-(4.4) we deduce that for $\langle a \rangle_p > \langle b \rangle_p + 1$,

$$\begin{aligned}
S_{p-1}(a, b) &= S_{p-1}(\langle a \rangle_p + pt, b) \equiv \left(\frac{b+1}{\langle a \rangle_p + pt} - 1 \right) S_{p-1}(\langle a \rangle_p + pt - 1, b) \\
&\equiv \left(\frac{b+1}{\langle a \rangle_p + pt} - 1 \right) \left(\frac{b+1}{\langle a \rangle_p - 1 + pt} - 1 \right) S_{p-1}(\langle a \rangle_p - 2 + pt, b) \\
&\equiv \cdots \equiv \prod_{k=\langle b \rangle_p + 2}^{\langle a \rangle_p} \left(\frac{b+1}{k + pt} - 1 \right) \cdot S_{p-1}(\langle b \rangle_p + 1 + pt, b) \\
&\equiv \prod_{k=\langle b \rangle_p + 2}^{\langle a \rangle_p} \frac{\langle b \rangle_p + 1 - k}{k} \cdot \frac{b - \langle b \rangle_p}{\langle b \rangle_p + 1} \\
&= \frac{(-1)^{\langle a \rangle_p - \langle b \rangle_p - 1}}{(\langle a \rangle_p - \langle b \rangle_p) \binom{\langle a \rangle_p}{\langle b \rangle_p}} (b - \langle b \rangle_p) \pmod{p^2}.
\end{aligned}$$

This completes the proof.

Corollary 4.1. *Let p be an odd prime and $a, b \in \mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{b-a}{k} \equiv \binom{\langle b \rangle_p}{\langle a \rangle_p} \pmod{p}.$$

Corollary 4.2. *Let p be an odd prime and $a \in \mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{-a}{k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } a \not\equiv 0 \pmod{p}, \\ 1 \pmod{p^2} & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

Proof. Taking $b = 0$ in Theorem 4.1 we deduce the result.

Corollary 4.3. *Let p be an odd prime and $a \in \mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k} \binom{1-a}{k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } a(1-a) \not\equiv 0 \pmod{p}, \\ 1+a \pmod{p^2} & \text{if } a \equiv 0 \pmod{p}, \\ 2-a \pmod{p^2} & \text{if } a \equiv 1 \pmod{p}. \end{cases}$$

Proof. Taking $b = 1$ in Theorem 4.1 we deduce the result.

We note that it can be easily proved that

$$(4.5) \quad \sum_{k=0}^n \binom{a}{k} \binom{-a}{k} = \binom{n+a}{n} \binom{n-a}{n},$$

$$(4.6) \quad \sum_{k=0}^n \binom{a}{k} \binom{1-a}{k} = -\frac{a^2 - a - n}{n^2} \binom{a-2}{n-1} \binom{-a-1}{n-1}.$$

Theorem 4.2. *Let p be an odd prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k} \\ & \equiv \begin{cases} (-1)^{\frac{p+1}{4}} p \left(\frac{\frac{p-1}{2}}{\frac{p+1}{4}} \right)^{-1} \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \\ (-1)^{\frac{p-1}{4}} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{4} \text{ and } 4 \mid x-1. \end{cases} \end{aligned}$$

Proof. Set $a = -\frac{1}{4}$ and $b = -\frac{3}{4}$. Then

$$\langle a \rangle_p = \begin{cases} \frac{p-1}{4} & \text{if } p \equiv 1 \pmod{4}, \\ \frac{3p-1}{4} & \text{if } p \equiv 3 \pmod{4}, \end{cases} \quad \text{and} \quad \langle b \rangle_p = \begin{cases} \frac{3p-3}{4} & \text{if } p \equiv 1 \pmod{4}, \\ \frac{p-3}{4} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

If $p \equiv 3 \pmod{4}$, then $\langle a \rangle_p > \langle b \rangle_p$. Thus, by Theorem 4.1 we have

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k} \equiv \frac{(-1)^{\frac{3p-1}{4} - \frac{p-3}{4} - 1}}{\left(\frac{3p-1}{4} - \frac{p-3}{4} \right) \left(\frac{3p-1}{4} \right)} \left(-\frac{3}{4} - \frac{p-3}{4} \right) \equiv \frac{3p}{2} \left(\frac{\frac{3(p+1)}{4}}{\frac{p+1}{4}} \right)^{-1} \pmod{p^2}.$$

Since

$$\begin{aligned} \left(\frac{\frac{3(p+1)}{4}}{\frac{p+1}{4}} \right) &= \frac{(p - \frac{p-3}{4})(p - \frac{p+1}{4}) \cdots (p - \frac{p-3}{2})}{\frac{p+1}{4}!} \\ &\equiv (-1)^{\frac{p+1}{4}} \frac{\frac{p-3}{2} \cdots \frac{p-3}{4}}{\frac{p+1}{4}!} = (-1)^{\frac{p+1}{4}} \left(\frac{\frac{p-1}{2}}{\frac{p+1}{4}} \right) \cdot \frac{(p-3)/4}{(p-1)/2} \\ &\equiv \frac{3}{2} (-1)^{\frac{p+1}{4}} \left(\frac{\frac{p-1}{2}}{\frac{p+1}{4}} \right) \pmod{p}, \end{aligned}$$

by the above we obtain the result in the case $p \equiv 3 \pmod{4}$.

Now we assume $p \equiv 1 \pmod{4}$ and so $p = x^2 + y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod{4}$. By the proof of Lemma 4.2 and Lemma 3.1(i) we have $H_{\frac{3(p-1)}{4}} \equiv H_{\frac{p-1}{4}} \equiv -3q_p(2) \pmod{p}$ and $H_{\frac{p-1}{2}} \equiv -2q_p(2)$. Now applying the above and Theorem 4.1 we deduce

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k} &\equiv \left(\frac{\frac{3(p-1)}{4}}{\frac{p-1}{4}} \right) \left(1 - \frac{3p}{4} H_{\frac{3(p-1)}{4}} + \frac{p}{4} H_{\frac{p-1}{4}} + \frac{p}{2} H_{\frac{p-1}{2}} \right) \\ &\equiv \left(\frac{\frac{3(p-1)}{4}}{\frac{p-1}{4}} \right) \left(1 - \frac{p}{2} (-3q_p(2)) + \frac{p}{2} (-2q_p(2)) \right) \\ &= \left(\frac{\frac{3(p-1)}{4}}{\frac{p-1}{4}} \right) \left(1 + \frac{1}{2} p q_p(2) \right) \pmod{p^2}. \end{aligned}$$

By [BEW, Theorem 9.4.3] we have

$$\left(\frac{\frac{3(p-1)}{4}}{\frac{p-1}{4}}\right) \equiv \left(2x - \frac{p}{2x}\right)(-1)^{\frac{p-1}{4}} \left(1 - \frac{1}{2}pq_p(2)\right) \pmod{p^2}.$$

Hence

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k} \equiv (-1)^{\frac{p-1}{4}} \left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

This proves the result in the case $p \equiv 1 \pmod{4}$. The proof is now complete.

Theorem 4.3. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-\frac{1}{6}}{k} \binom{-\frac{1}{3}}{k} \\ & \equiv \begin{cases} \frac{3p}{2} \left(\frac{p-1}{\frac{p-5}{6}}\right)^{-1} \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \\ 2A - \frac{p}{2A} \pmod{p^2} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3} \text{ and } 3 \mid A-1. \end{cases} \end{aligned}$$

Proof. Set $a = -\frac{1}{6}$ and $b = -\frac{1}{2}$. Then

$$\langle b \rangle_p = \frac{p-1}{2} \quad \text{and} \quad \langle a \rangle_p = \begin{cases} \frac{p-1}{6} & \text{if } p \equiv 1 \pmod{3}, \\ \frac{5p-1}{6} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

For $p \equiv 2 \pmod{3}$ we have $\langle a \rangle_p > \langle b \rangle_p$. Thus, by Theorem 4.1 we get

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-\frac{1}{6}}{k} \binom{-\frac{1}{3}}{k} \\ & \equiv \frac{(-1)^{\frac{5p-1}{6} - \frac{p-1}{2} - 1}}{\left(\frac{5p-1}{6} - \frac{p-1}{2}\right) \binom{\frac{5p-1}{6}}{\frac{p-1}{2}}} \left(-\frac{1}{2} - \frac{p-1}{2}\right) = \frac{3p}{2(p+1) \binom{\frac{5p-1}{6}}{\frac{p-1}{2}}} \pmod{p^2}. \end{aligned}$$

Note that

$$\begin{aligned} \left(\frac{\frac{5p-1}{6}}{\frac{p-1}{2}}\right) &= \left(\frac{\frac{5p-1}{6}}{\frac{p+1}{3}}\right) = \left(p - \frac{p+1}{6}\right) \equiv \left(-\frac{p+1}{6}\right) \\ &= \left(\frac{p+1}{6} + \frac{p+1}{3} - 1\right) = \left(\frac{p-1}{2}\right) = \left(\frac{p-1}{\frac{p-5}{6}}\right) \pmod{p}. \end{aligned}$$

We then get the result in the case $p \equiv 2 \pmod{3}$.

Now we assume $p \equiv 1 \pmod{3}$ and so $p = A^2 + 3B^2 \equiv 1 \pmod{3}$ with $3 \mid A - 1$. By Theorem 4.1 and Lemma 3.1 we obtain

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{-\frac{1}{6}}{k} \binom{-\frac{1}{3}}{k} \\
& \equiv \left(\frac{\frac{p-1}{2}}{\frac{p-1}{6}} \right) \left(1 - \frac{p}{2} H_{\frac{p-1}{2}} + \frac{p}{6} H_{\frac{p-1}{6}} + \frac{p}{3} H_{\frac{p-1}{3}} \right) \\
& \equiv \left(\frac{\frac{p-1}{2}}{\frac{p-1}{6}} \right) \left(1 - \frac{p}{2} (-2q_p(2)) + \frac{p}{6} (-2q_p(2) - \frac{3}{2}q_p(3)) + \frac{p}{3} (-\frac{3}{2}q_p(3)) \right) \\
& = \left(\frac{\frac{p-1}{2}}{\frac{p-1}{6}} \right) \left(1 + p \left(\frac{2}{3}q_p(2) - \frac{3}{4}q_p(3) \right) \right) \pmod{p^2}.
\end{aligned}$$

By [BEW, Theorem 9.4.4],

$$\left(\frac{\frac{p-1}{2}}{\frac{p-1}{6}} \right) \equiv \left(2A - \frac{p}{2A} \right) \left(1 - p \left(\frac{2}{3}q_p(2) - \frac{3}{4}q_p(3) \right) \right) \pmod{p^2}.$$

Therefore,

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{-\frac{1}{6}}{k} \binom{-\frac{1}{3}}{k} \\
& \equiv \left(2A - \frac{p}{2A} \right) \left(1^2 - p^2 \left(\frac{2}{3}q_p(2) - \frac{3}{4}q_p(3) \right)^2 \right) \equiv 2A - \frac{p}{2A} \pmod{p^2}.
\end{aligned}$$

This proves the case $p \equiv 1 \pmod{3}$. Hence the theorem is proved.

Conjecture 4.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k}}{4^k} \equiv \begin{cases} 2x - \frac{p}{2x} \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{12} \text{ and } 2 \nmid x, \\ 2y - \frac{p}{2y} \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 5 \pmod{12} \text{ and } 2 \nmid x, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Conjecture 4.2. *Let p be an odd prime. Then*

$$\begin{aligned}
& \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k}}{(-3)^k} \equiv (-1)^{\frac{p-1}{4}} \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k}}{81^k} \\
& \equiv \begin{cases} 2x - \frac{p}{2x} \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{4} \text{ and } 2 \nmid x, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}
\end{aligned}$$

Conjecture 4.3. *Let p be an odd prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k}}{(-80)^k} \\ & \equiv \begin{cases} 2x - \frac{p}{2x} \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv \pm 1 \pmod{5} \text{ and } 2 \nmid x, \\ 2y - \frac{p}{2y} \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv \pm 2 \pmod{5} \text{ and } 2 \nmid x, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

Conjecture 4.4. *Let $p > 5$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k} 2^k \\ & = \begin{cases} 2x - \frac{p}{2x} \pmod{p^2} & \text{if } p = x^2 + 2y^2 \text{ with } x \equiv 1 \pmod{4}, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

Conjecture 4.5. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{6}}{k} \binom{-\frac{1}{3}}{k}}{2^k} \\ & \equiv \begin{cases} \left(\frac{x}{3}\right)(2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ \left(\frac{x}{3}\right)(2x - \frac{p}{4x}) \pmod{p^2} & \text{if } p = 2x^2 + 3y^2 \equiv 5, 11 \pmod{24}, \\ 0 \pmod{p} & \text{if } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases} \end{aligned}$$

Conjecture 4.6. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{6}}{k} \binom{-\frac{1}{3}}{k}}{(-16)^k} \equiv \begin{cases} -\left(\frac{x}{3}\right)(x - \frac{p}{x}) \pmod{p^2} & \text{if } 4p = x^2 + 51y^2, \\ 0 \pmod{p} & \text{if } \left(\frac{p}{51}\right) = -1. \end{cases}$$

Conjecture 4.7. *Let p be a prime of the form $6k + 5$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{6}}{k} \binom{-\frac{1}{3}}{k}}{(-80)^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{6}}{k} \binom{-\frac{1}{3}}{k}}{(-3024)^k} \equiv 0 \pmod{p}.$$

Conjecture 4.8. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{6}}{k} \binom{-\frac{1}{3}}{k}}{(-1024)^k} \equiv \begin{cases} -\left(\frac{x}{3}\right)(x - \frac{p}{x}) \pmod{p^2} & \text{if } 4p = x^2 + 123y^2, \\ 0 \pmod{p} & \text{if } \left(\frac{p}{123}\right) = -1. \end{cases}$$

Conjecture 4.9. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{6}}{k} \binom{-\frac{1}{3}}{k}}{(-250000)^k} \equiv \begin{cases} -(\frac{x}{3})(x - \frac{p}{x}) \pmod{p^2} & \text{if } 4p = x^2 + 267y^2, \\ 0 \pmod{p} & \text{if } (\frac{p}{267}) = -1. \end{cases}$$

Conjecture 4.10. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k} (-4)^k = \begin{cases} (\frac{x}{3})(2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + 15y^2, \\ -(\frac{x}{3})(10x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = 5x^2 + 3y^2, \\ 0 \pmod{p} & \text{if } (\frac{p}{15}) = -1. \end{cases}$$

Conjecture 4.11. *Let $p > 3$ be a prime such that $p \equiv 13, 17, 19, 23 \pmod{24}$. Then*

$$\sum_{k=0}^{p-1} (-1)^k \binom{-\frac{1}{6}}{k} \binom{-\frac{2}{3}}{k} \equiv 0 \pmod{p}.$$

Conjecture 4.12. *Let p be a prime such that $p \equiv 5 \pmod{6}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{6}}{k} \binom{-\frac{2}{3}}{k}}{9^{2k}} \equiv \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{6}}{k} \binom{-\frac{2}{3}}{k}}{55^{2k}} \equiv 0 \pmod{p}.$$

Conjecture 4.13. *Let $p > 5$ be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k} (-3)^k &\equiv \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k}}{(-27)^k} \\ &\equiv \left(\frac{p}{5}\right) \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k}}{5^k} \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k} 2^k \\ &\equiv \begin{cases} 2A - \frac{p}{2A} \pmod{p^2} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3} \text{ with } 3 \mid A - 1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Conjecture 4.14. *Let $p > 5$ be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k}}{(-4)^k} \\ = \begin{cases} \left(\frac{p}{5}\right)(2A - \frac{p}{2A}) \pmod{p^2} \\ \quad \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3} \text{ with } 5 \mid AB \text{ and } 3 \mid A - 1, \\ \left(\frac{p}{5}\right)(A + 3B - \frac{p}{A+3B}) \pmod{p^2} \\ \quad \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3} \text{ with } A/B \equiv -1, -2 \pmod{5} \text{ and } 3 \mid A - 1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Conjecture 4.15. *Let $p \neq 2, 5$ be a prime such that $(\frac{-5}{p}) = -1$. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{8}}{k} \binom{-\frac{5}{8}}{k} \frac{1}{5^k} \equiv \sum_{k=0}^{p-1} \binom{-\frac{3}{8}}{k} \binom{-\frac{7}{8}}{k} \frac{1}{5^k} \equiv 0 \pmod{p}.$$

Conjecture 4.16. *Let $p \neq 7$ be an odd prime such that $(\frac{-1}{p}) = -1$. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{8}}{k} \binom{-\frac{5}{8}}{k} \frac{1}{49^k} \equiv \sum_{k=0}^{p-1} \binom{-\frac{3}{8}}{k} \binom{-\frac{7}{8}}{k} \frac{1}{49^k} \equiv 0 \pmod{p}.$$

Conjecture 4.17. *Let $p \neq 2, 3$ be a prime such that $(\frac{-6}{p}) = -1$. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{8}}{k} \binom{-\frac{5}{8}}{k} \frac{1}{(-8)^k} \equiv \sum_{k=0}^{p-1} \binom{-\frac{3}{8}}{k} \binom{-\frac{7}{8}}{k} \frac{1}{(-8)^k} \equiv 0 \pmod{p}.$$

Conjecture 4.18. *Let $p > 5$ be a prime such that $(\frac{-2}{p}) = -1$. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{8}}{k} \binom{-\frac{5}{8}}{k} \frac{1}{(-2400)^k} \equiv \sum_{k=0}^{p-1} \binom{-\frac{3}{8}}{k} \binom{-\frac{7}{8}}{k} \frac{1}{(-2400)^k} \equiv 0 \pmod{p}.$$

Conjecture 4.19. *Let $p \neq 2, 5$ be a prime such that $(\frac{-10}{p}) = -1$. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{8}}{k} \binom{-\frac{5}{8}}{k} \frac{1}{(-80)^k} \equiv \sum_{k=0}^{p-1} \binom{-\frac{3}{8}}{k} \binom{-\frac{7}{8}}{k} \frac{1}{(-80)^k} \equiv 0 \pmod{p}.$$

Conjecture 4.20. *Let $p \neq 2, 3, 19$ be a prime with $(\frac{-19}{p}) = -1$. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{12}}{k} \binom{-\frac{7}{12}}{k} \frac{1}{513^k} \equiv 0 \pmod{p}.$$

Conjecture 4.21. *Let $p \neq 2, 3, 17$ be a prime with $(\frac{-51}{p}) = -1$. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{5}{6}}{k} \frac{1}{17^k} \equiv 0 \pmod{p}.$$

Conjecture 4.22. Let $p \neq 2, 3$ be a prime with $(\frac{-3}{p}) = -1$. Then

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{5}{6}}{k} \frac{1}{81^k} \equiv 0 \pmod{p}.$$

Conjecture 4.23. Let $p \neq 2, 3$ be a prime with $(\frac{-6}{p}) = -1$. Then

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{5}{6}}{k} (-1)^k \equiv 0 \pmod{p}.$$

5. Congruences for $\sum_{k=0}^{p-1} (\frac{-1}{k})^2 m^k$ and $\sum_{k=0}^{p-1} (\frac{-1}{k})^2 m^k$.

Theorem 5.1. Let p be an odd prime. Then

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 \\ & \equiv \begin{cases} 2p \left(\frac{p-1}{\frac{p+1}{4}} \right)^{-1} \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \\ 2x - \frac{p}{2x} \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{4} \text{ and } 4 \mid x-1. \end{cases} \end{aligned}$$

Proof. Set $a = -\frac{1}{4}$ and $b = -\frac{1}{2}$. Then $\langle b \rangle_p = \frac{p-1}{2}$, $b - a = -\frac{1}{4}$ and

$$\langle a \rangle_p = \langle b - a \rangle_p = \begin{cases} \frac{p-1}{4} & \text{if } p \equiv 1 \pmod{4}, \\ \frac{3p-1}{4} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

If $p \equiv 3 \pmod{4}$, then $\langle a \rangle_p > \langle b \rangle_p$. Thus, by Theorem 4.1 we have

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 \equiv \frac{(-1)^{\frac{3p-1}{4} - \frac{p-1}{2} - 1}}{(\frac{3p-1}{4} - \frac{p-1}{2})(\frac{3p-1}{2})} \left(-\frac{p}{2} \right) \equiv 2p(-1)^{\frac{p+1}{4}} \left(\frac{3p-1}{4} \right)^{-1} \pmod{p^2}.$$

Since

$$\begin{aligned} \left(\frac{\frac{3p-1}{4}}{\frac{p-1}{2}} \right) &= \left(\frac{\frac{3p-1}{4}}{\frac{p+1}{4}} \right) = \left(p - \frac{p+1}{4} \right) \equiv \left(-\frac{p+1}{4} \right) \\ &= (-1)^{\frac{p+1}{4}} \left(\frac{\frac{p+1}{4}}{\frac{p+1}{4}} - 1 \right) = (-1)^{\frac{p+1}{4}} \left(\frac{\frac{p-1}{2}}{\frac{p+1}{4}} \right) \pmod{p}, \end{aligned}$$

by the above we obtain the result in the case $p \equiv 3 \pmod{4}$.

Now we assume $p \equiv 1 \pmod{4}$ and so $p = x^2 + y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod{4}$. By Lemma 3.1(i) we have $H_{\frac{p-1}{4}} \equiv -3q_p(2) \pmod{p}$ and $H_{\frac{p-1}{2}} \equiv -2q_p(2)$. Now applying the above and Theorem 4.1 we deduce

$$\begin{aligned} \sum_{k=0}^{p-1} \left(-\frac{1}{4} \right)_k^2 &\equiv \left(\frac{\frac{p-1}{2}}{\frac{p-1}{4}} \right) \left(1 - \frac{p}{2} H_{\frac{p-1}{2}} + \frac{p}{2} H_{\frac{p-1}{4}} \right) \\ &\equiv \left(\frac{\frac{p-1}{2}}{\frac{p-1}{4}} \right) \left(1 - \frac{p}{2} (-2q_p(2)) + \frac{p}{2} (-3q_p(2)) \right) \\ &= \left(\frac{\frac{p-1}{2}}{\frac{p-1}{4}} \right) \left(1 - \frac{1}{2} p q_p(2) \right) \pmod{p^2}. \end{aligned}$$

By [BEW] we have

$$\left(\frac{\frac{p-1}{2}}{\frac{p-1}{4}} \right) \equiv \left(2x - \frac{p}{2x} \right) \left(1 + \frac{1}{2} p q_p(2) \right) \pmod{p^2}.$$

Hence

$$\sum_{k=0}^{p-1} \left(-\frac{1}{4} \right)_k^2 \equiv 2x - \frac{p}{2x} \pmod{p^2}.$$

This proves the result in the case $p \equiv 1 \pmod{4}$. The proof is now complete.

Remark 5.1 Let p be an odd prime and $a \in \mathbb{Z}_p$. From [Su4] we know that

$$\sum_{k=0}^{p-1} \binom{a}{k}^2 \equiv \binom{2a}{\langle a \rangle_p} \pmod{p^2}.$$

Hence

$$\sum_{k=0}^{p-1} \left(-\frac{1}{4} \right)_k^2 \equiv \begin{cases} \left(\frac{\frac{p-1}{2}}{\frac{p-1}{4}} \right) \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ \left(\frac{\frac{p-1}{2}}{\frac{3p-1}{4}} \right) \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Lemma 5.1 ([G, (3.134)]). *Let n be a nonnegative integer and $x \neq 1$. Then*

$$P_n(x) = \left(\frac{x-1}{2} \right)^n \sum_{k=0}^n \binom{n}{k}^2 \left(\frac{x+1}{x-1} \right)^k.$$

Lemma 5.2. *Let p be an odd prime and $a \in \mathbb{Z}_p$ with $a \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k}^2 t^k \equiv t^{\langle a \rangle_p} \sum_{k=0}^{p-1} \frac{\binom{a}{k}^2}{t^k} \pmod{p}$$

and

$$\sum_{k=0}^{p-1} \binom{a}{k}^2 t^k \equiv (t-1)^{\langle a \rangle_p} P_{\langle a \rangle_p} \left(\frac{t+1}{t-1} \right) \pmod{p}.$$

Proof. It is clear that

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{a}{k}^2 t^k &\equiv \sum_{k=0}^{\langle a \rangle_p} \binom{\langle a \rangle_p}{k}^2 t^k = \sum_{s=0}^{\langle a \rangle_p} \binom{\langle a \rangle_p}{\langle a \rangle_p - s}^2 t^{\langle a \rangle_p - s} \\ &= t^{\langle a \rangle_p} \sum_{k=0}^{\langle a \rangle_p} \binom{\langle a \rangle_p}{k}^2 t^{-k} \equiv t^{\langle a \rangle_p} \sum_{k=0}^{p-1} \frac{\binom{a}{k}^2}{t^k} \pmod{p}. \end{aligned}$$

Using Lemma 5.1 we see that for $x \neq 1$,

$$\begin{aligned} P_{\langle a \rangle_p}(x) &= \left(\frac{x-1}{2} \right)^{\langle a \rangle_p} \sum_{k=0}^{\langle a \rangle_p} \binom{\langle a \rangle_p}{k}^2 \left(\frac{x+1}{x-1} \right)^k \\ &= \left(\frac{x-1}{2} \right)^{\langle a \rangle_p} \sum_{k=0}^{p-1} \binom{\langle a \rangle_p}{k}^2 \left(\frac{x+1}{x-1} \right)^k \\ &\equiv \left(\frac{x-1}{2} \right)^{\langle a \rangle_p} \sum_{k=0}^{p-1} \binom{a}{k}^2 \left(\frac{x+1}{x-1} \right)^k \pmod{p}. \end{aligned}$$

Set $x = \frac{t+1}{t-1}$. Then $t = \frac{x+1}{x-1}$ and $\frac{x-1}{2} = \frac{1}{t-1}$. Now substituting x with $\frac{t+1}{t-1}$ in the above congruence we obtain the remaining result.

Lemma 5.3. *Let p be an odd prime and $m \in \{1, 2, \dots, \frac{p-1}{2}\}$. Then $P_{p-1-m}(x) \equiv P_m(x)$.*

Proof. Since $m < \frac{p}{2}$ we have $p-1-m \geq m$. It is well-known that $\binom{-t}{k} = (-1)^k \binom{t+k-1}{k}$ and

$$P_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \left(\frac{x-1}{2} \right)^k.$$

Thus,

$$\begin{aligned}
P_{p-1-m}(x) &= \sum_{k=0}^{p-1-m} \binom{p-1-m}{k} \binom{p-1-m+k}{k} \left(\frac{x-1}{2}\right)^k \\
&\equiv \sum_{k=0}^{p-1-m} \binom{-1-m}{k} \binom{-1-m+k}{k} \left(\frac{x-1}{2}\right)^k \\
&= \sum_{k=0}^{p-1-m} (-1)^k \binom{m+k}{k} \cdot (-1)^k \binom{m}{k} \left(\frac{x-1}{2}\right)^k \\
&= \sum_{k=0}^m \binom{m}{k} \binom{m+k}{k} \left(\frac{x-1}{2}\right)^k = P_m(x) \pmod{p}.
\end{aligned}$$

This proves the lemma.

Theorem 5.2. *Let p be an odd prime and $a, t \in \mathbb{Z}_p$ with $a \not\equiv -1 \pmod{p}$ and $t \not\equiv 1 \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} \binom{-1-a}{k}^2 t^k \equiv (t-1)^{-2\langle a \rangle_p} \sum_{k=0}^{p-1} \binom{a}{k}^2 t^k \pmod{p}.$$

Proof. Clearly $\langle -1-a \rangle_p = p-1-\langle a \rangle_p$. Thus, using Lemmas 5.2 and 5.3 we see that for $t \not\equiv 1 \pmod{p}$,

$$\begin{aligned}
&\sum_{k=0}^{p-1} \binom{-1-a}{k}^2 t^k \\
&\equiv (t-1)^{\langle -1-a \rangle_p} P_{\langle -1-a \rangle_p} \left(\frac{t+1}{t-1} \right) = (t-1)^{p-1-\langle a \rangle_p} P_{p-1-\langle a \rangle_p} \left(\frac{t+1}{t-1} \right) \\
&\equiv (t-1)^{-\langle a \rangle_p} P_{\langle a \rangle_p} \left(\frac{t+1}{t-1} \right) \equiv (t-1)^{-2\langle a \rangle_p} \sum_{k=0}^{p-1} \binom{a}{k}^2 t^k \pmod{p}.
\end{aligned}$$

This proves the theorem.

Theorem 5.3. *Let $p > 3$ be a prime. Then*

$$\begin{aligned}
&\sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k}^2 9^k \equiv \frac{1}{3^{\lfloor \frac{p}{3} \rfloor}} \sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k}^2 \frac{1}{9^k} \\
&\equiv \begin{cases} L \pmod{p} & \text{if } p \equiv 1 \pmod{3}, 4p = L^2 + 27M^2 \text{ and } 3 \mid L-2, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}
\end{aligned}$$

Proof. By Lemmas 5.2 and 5.3 we have

$$\begin{aligned}
& 9^{\langle -\frac{1}{3} \rangle_p} \sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k}^2 \frac{1}{9^k} \\
& \equiv \sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k}^2 9^k \equiv 8^{\langle -\frac{1}{3} \rangle_p} P_{\langle -\frac{1}{3} \rangle_p} \left(\frac{10}{8} \right) \\
& = \begin{cases} 8^{\frac{p-1}{3}} P_{\frac{p-1}{3}} \left(\frac{5}{4} \right) \equiv P_{\frac{p-1}{3}} \left(\frac{5}{4} \right) \pmod{p} & \text{if } p \equiv 1 \pmod{3}, \\ 8^{\frac{2p-1}{3}} P_{\frac{2p-1}{3}} \left(\frac{5}{4} \right) \equiv 2P_{\frac{p-2}{3}} \left(\frac{5}{4} \right) \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}
\end{aligned}$$

From [S4, Theorem 3.2] we know that

$$P_{[\frac{p}{3}]} \left(\frac{5}{4} \right) \equiv \begin{cases} L \pmod{p} & \text{if } p \equiv 1 \pmod{3}, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Since

$$9^{\langle -\frac{1}{3} \rangle_p} = \begin{cases} 9^{\frac{p-1}{3}} \equiv 3^{-\frac{p-1}{3}} \pmod{p} & \text{if } p \equiv 1 \pmod{3}, \\ 9^{\frac{2p-1}{3}} \equiv 3^{-\frac{p-2}{3}} \pmod{p} & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

combining all the above we deduce the result.

Conjecture 5.1. *Let p be a prime such that $p \equiv 1 \pmod{3}$, $4p = L^2 + 27M^2$ ($L, M \in \mathbb{Z}$) and $L \equiv 2 \pmod{3}$. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k}^2 9^k \equiv L - \frac{p}{L} \pmod{p^2}$$

Theorem 5.4. *Let p be an odd prime. Then*

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 (-8)^k \\
& \equiv \begin{cases} (-1)^{\frac{p-1}{4}} 2x \pmod{p} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{4} \text{ and } 4 \mid x-1, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 \frac{1}{(-8)^k} \\
& \equiv \begin{cases} (-1)^{\frac{y}{4}} 2x \pmod{p} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{8} \text{ and } 4 \mid x-1, \\ (-1)^{\frac{y-2}{4}} 2y \pmod{p} & \text{if } p = x^2 + y^2 \equiv 5 \pmod{8} \text{ and } 2 \nmid x, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}
\end{aligned}$$

Proof. By Lemmas 5.2 and 5.3 we have

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 (-8)^k \\
& \equiv (-8)^{\langle -\frac{1}{4} \rangle_p} \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 \frac{1}{(-8)^k} \\
& \equiv (-9)^{\langle -\frac{1}{4} \rangle_p} P_{\langle -\frac{1}{4} \rangle_p} \left(\frac{7}{9} \right) = 9^{\langle -\frac{1}{4} \rangle_p} P_{\langle -\frac{1}{4} \rangle_p} \left(-\frac{7}{9} \right) \\
& = \begin{cases} 9^{\frac{p-1}{4}} P_{\frac{p-1}{4}} \left(-\frac{7}{9} \right) \equiv \left(\frac{3}{p} \right) P_{\frac{p-1}{4}} \left(-\frac{7}{9} \right) \pmod{p} & \text{if } p \equiv 1 \pmod{4}, \\ 9^{\frac{3p-1}{4}} P_{\frac{3p-1}{4}} \left(-\frac{7}{9} \right) \equiv 9^{\frac{3p-1}{4}} P_{\frac{p-3}{4}} \left(-\frac{7}{9} \right) \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}
\end{aligned}$$

From [S4, Theorem 2.4] we know that

$$P_{[\frac{p}{4}]} \left(-\frac{7}{9} \right) \equiv \begin{cases} (-1)^{\frac{p-1}{4}} \left(\frac{p}{3} \right) 2x \pmod{p} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{4} \text{ and } 4 \mid x-1, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

When $p \equiv 1 \pmod{4}$, we have $(-8)^{-\langle -\frac{1}{4} \rangle_p} = (-8)^{-\frac{p-1}{4}} \equiv (-2)^{\frac{p-1}{4}} \pmod{p}$. It is well known that (see [BEW])

$$2^{\frac{p-1}{4}} \equiv \begin{cases} (-1)^{\frac{y}{4}} \pmod{p} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{\frac{y-2}{4}} \frac{y}{x} \pmod{p} & \text{if } p \equiv 5 \pmod{8}. \end{cases}$$

Now combining all the above we deduce the result.

Conjecture 5.2. *Let p be an odd prime. Then*

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 (-8)^k \\
& \equiv \begin{cases} (-1)^{\frac{p-1}{4}} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{4} \text{ and } 4 \mid x-1, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k}^2}{(-8)^k} \\
& \equiv \begin{cases} (-1)^{\frac{y}{4}} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{8} \text{ and } 4 \mid x-1, \\ (-1)^{\frac{y-2}{4}} (2y - \frac{p}{2y}) \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 5 \pmod{8} \text{ and } 4 \mid y-2, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}
\end{aligned}$$

Theorem 5.5. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 4^k &\equiv \frac{3 - (-1)^{\frac{p-1}{2}}}{2} \left(\frac{2}{p}\right) \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 \frac{1}{4^k} \\ &\equiv \begin{cases} (-1)^{\frac{p-1}{4} + \frac{A-1}{2}} 2A \pmod{p} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{12}, \\ (-1)^{\frac{p+1}{4}} 6B \pmod{p} & \text{if } p = A^2 + 3B^2 \equiv 7 \pmod{12} \text{ and } 4 \mid B-1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3} \end{cases} \end{aligned}$$

Proof. By Lemmas 5.2 and 5.3 we have

$$\begin{aligned} &\sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 4^k \\ &\equiv 4^{\langle -\frac{1}{4} \rangle_p} \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 \frac{1}{4^k} \\ &\equiv 3^{\langle -\frac{1}{4} \rangle_p} P_{\langle -\frac{1}{4} \rangle_p} \left(\frac{5}{3} \right) = (-3)^{\langle -\frac{1}{4} \rangle_p} P_{\langle -\frac{1}{4} \rangle_p} \left(-\frac{5}{3} \right) \\ &= \begin{cases} (-3)^{\frac{p-1}{4}} P_{\frac{p-1}{4}} \left(-\frac{5}{3} \right) \pmod{p} & \text{if } p \equiv 1 \pmod{4}, \\ (-3)^{\frac{3p-1}{4}} P_{\frac{3p-1}{4}} \left(-\frac{5}{3} \right) \equiv (-3)^{-\frac{p-3}{4}} P_{\frac{p-3}{4}} \left(-\frac{5}{3} \right) \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

From [S4, Theorem 2.5] we know that

$$P_{[\frac{p}{4}]} \left(-\frac{5}{3} \right) \equiv \begin{cases} 2A \pmod{p} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3} \text{ and } 3 \mid A-1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Hence the result is true for $p \equiv 2 \pmod{3}$.

Now assume $p = A^2 + 3B^2 \equiv 1 \pmod{3}$ and $A \equiv 1 \pmod{3}$. If $p \equiv 1 \pmod{12}$, by [S5, p.1317] we have $3^{\frac{p-1}{4}} \equiv (-1)^{\frac{A-1}{2}} \pmod{p}$ and $4^{\langle -\frac{1}{4} \rangle_p} = 4^{\frac{p-1}{4}} \equiv \left(\frac{2}{p}\right) = (-1)^{\frac{p-1}{4}} \pmod{p}$. Hence

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 4^k \equiv (-1)^{\frac{p-1}{4}} \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 \frac{1}{4^k} \equiv (-1)^{\frac{p-1}{4} + \frac{A-1}{2}} 2A \pmod{p}.$$

If $p \equiv 7 \pmod{12}$ and $B \equiv 1 \pmod{4}$, by [S5, p.1317] we have $3^{\frac{p-3}{4}} \equiv \frac{B}{A} \pmod{p}$. Since $4^{\langle -\frac{1}{4} \rangle_p} = 4^{\frac{3p-1}{4}} = 2^{p-1+\frac{p+1}{2}} \equiv 2\left(\frac{2}{p}\right) \pmod{p}$, by the above we get

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 4^k \equiv 2\left(\frac{2}{p}\right) \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 \frac{1}{4^k} \equiv (-1)^{\frac{p-3}{4}} \frac{A}{B} \cdot 2A \equiv (-1)^{\frac{p+1}{4}} 6B \pmod{p}.$$

Now combining all the above we deduce the result.

Conjecture 5.3. *Let $p > 3$ be an odd prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 4^k &\equiv \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k} (-8)^k \\ &\equiv \begin{cases} (-1)^{\frac{p-1}{4} + \frac{A-1}{2}} (2A - \frac{p}{2A}) \pmod{p^2} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{12}, \\ (-1)^{\frac{p+1}{4} + \frac{B-1}{2}} (6B - \frac{p}{2B}) \pmod{p^2} & \text{if } p = A^2 + 3B^2 \equiv 7 \pmod{12}, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3} \end{cases} \end{aligned}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k}^2}{4^k} \equiv \begin{cases} (-1)^{\frac{A-1}{2}} (2A - \frac{p}{2A}) \pmod{p^2} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{12}, \\ (-1)^{\frac{B-1}{2}} (3B - \frac{p}{4B}) \pmod{p^2} & \text{if } p = A^2 + 3B^2 \equiv 7 \pmod{12}, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Theorem 5.6. *Let $p \neq 2, 7$ be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 64^k &\equiv \frac{9 - 7(-1)^{\frac{p-1}{2}}}{2} \left(\frac{2}{p}\right) \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 \frac{1}{64^k} \\ &\equiv \begin{cases} (-1)^{\frac{p-1}{4} + \frac{x-1}{2}} 2x \pmod{p} & \text{if } p = x^2 + 7y^2 \equiv 1 \pmod{4}, \\ (-1)^{\frac{p+1}{4} + \frac{y-1}{2}} 42y \pmod{p} & \text{if } p = x^2 + 7y^2 \equiv 3 \pmod{4}, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases} \end{aligned}$$

Proof. By Lemmas 5.2 and 5.3 we have

$$\begin{aligned} &\sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 64^k \\ &\equiv 64^{\langle -\frac{1}{4} \rangle_p} \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 \frac{1}{64^k} \\ &\equiv 63^{\langle -\frac{1}{4} \rangle_p} P_{\langle -\frac{1}{4} \rangle_p} \left(\frac{65}{63} \right) = (-63)^{\langle -\frac{1}{4} \rangle_p} P_{\langle -\frac{1}{4} \rangle_p} \left(-\frac{65}{63} \right) \\ &= \begin{cases} (-63)^{\frac{p-1}{4}} P_{\frac{p-1}{4}} \left(-\frac{65}{63} \right) \pmod{p} & \text{if } p \equiv 1 \pmod{4}, \\ (-63)^{\frac{3p-1}{4}} P_{\frac{3p-1}{4}} \left(-\frac{65}{63} \right) \equiv (-63)^{-\frac{p-3}{4}} P_{\frac{p-3}{4}} \left(-\frac{65}{63} \right) \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

By [S4, Theorem 2.6],

$$P_{\left[\frac{p}{4}\right]} \left(-\frac{65}{63} \right) \equiv \begin{cases} 2x \left(\frac{p}{3} \right) \left(\frac{x}{7} \right) \pmod{p} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod{7}, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

Hence the result is true for $p \equiv 3, 5, 6 \pmod{7}$.

Now suppose $p \equiv 1, 2, 4 \pmod{7}$ and so $p = x^2 + 7y^2$ with $x, y \in \mathbb{Z}$. If $p \equiv 1 \pmod{4}$, by [S5, p.1317] we have $7^{\frac{p-1}{4}} \equiv (-1)^{\frac{x-1}{2}} \left(\frac{x}{7}\right) \pmod{p}$ and so

$$\begin{aligned} & (-63)^{\frac{p-1}{4}} P_{\frac{p-1}{4}} \left(-\frac{65}{63}\right) \\ & \equiv (-1)^{\frac{p-1}{4}} \left(\frac{3}{p}\right) \cdot (-1)^{\frac{x-1}{2}} \left(\frac{x}{7}\right) \cdot 2x \left(\frac{p}{3}\right) \left(\frac{x}{7}\right) = (-1)^{\frac{p-1}{4} + \frac{x-1}{2}} 2x \pmod{p}. \end{aligned}$$

If $p \equiv 3 \pmod{4}$, by [S5, p.1317] we have $7^{\frac{p-3}{4}} \equiv (-1)^{\frac{y+1}{2}} \left(\frac{x}{7}\right) \frac{y}{x} \pmod{p}$ and so

$$\begin{aligned} & (-63)^{-\frac{p-3}{4}} P_{\frac{p-3}{4}} \left(-\frac{65}{63}\right) \\ & \equiv (-1)^{\frac{p-3}{4}} 3 \left(\frac{3}{p}\right) \cdot (-1)^{\frac{y+1}{2}} \left(\frac{x}{7}\right) \frac{x}{y} \cdot 2x \left(\frac{p}{3}\right) \left(\frac{x}{7}\right) \equiv (-1)^{\frac{p+1}{4} + \frac{y-1}{2}} 42y \pmod{p}. \end{aligned}$$

Note that

$$64^{\langle -\frac{1}{4} \rangle_p} = \begin{cases} 64^{\frac{p-1}{4}} \equiv \left(\frac{2}{p}\right) \pmod{p} & \text{if } p \equiv 1 \pmod{4}, \\ 64^{\frac{3p-1}{4}} \equiv 8 \left(\frac{2}{p}\right) \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Combining all the above we deduce the result.

Conjecture 5.4. *Let $p \neq 2, 7$ be a prime. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 64^k \equiv \begin{cases} \left(\frac{2}{p}\right) (-1)^{\frac{x-1}{2}} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + 7y^2 \equiv 1 \pmod{4}, \\ \left(\frac{2}{p}\right) (-1)^{\frac{y-1}{2}} (42y - \frac{3p}{2y}) \pmod{p^2} & \text{if } p = x^2 + 7y^2 \equiv 3 \pmod{4}, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k}^2}{64^k} \equiv \begin{cases} (-1)^{\frac{x-1}{2}} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + 7y^2 \equiv 1 \pmod{4}, \\ \frac{3}{4} (-1)^{\frac{y-1}{2}} (7y - \frac{p}{4y}) \pmod{p^2} & \text{if } p = x^2 + 7y^2 \equiv 3 \pmod{4}, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

Conjecture 5.5. *Let p be a prime such that $p \equiv 5, 7 \pmod{8}$. Then*

$$\sum_{k=0}^{p-1} (-1)^k \binom{-\frac{1}{4}}{k}^2 \equiv \begin{cases} (-1)^{\frac{x+1}{2}} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 1 \pmod{8}, \\ (-1)^{\frac{y-1}{2}} (4y - \frac{p}{2y}) \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 3 \pmod{8}, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

REFERENCES

- [B] H. Bateman, *Higher transcendental functions, Vol.I*, McGraw-Hill Book Co. Inc., 1953.
- [Be] F. Beukers, *Another congruence for the Apéry numbers*, J. Number Theory **25** (1987), 201-210.
- [BEW] B.C. Berndt, R.J. Evans and K.S. Williams, *Gauss and Jacobi Sums*, John Wiley & Sons, New York, 1998.
- [G] H.W. Gould, *Combinatorial Identities, A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*, Morgantown, W. Va., 1972.
- [L] E. Lehmer, *On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson*, Ann. of Math. **39** (1938), 350-360.
- [M1] E. Mortenson, *A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function*, J. Number Theory **99** (2003), 139-147.
- [M2] E. Mortenson, *Supercongruences between truncated ${}_2F_1$ by hypergeometric functions and their Gaussian analogs*, Trans. Amer. Math. Soc. **355** (2003), 987-1007.
- [M3] E. Mortenson, *Supercongruences for truncated ${}_{n+1}F_n$ hypergeometric series with applications to certain weight three newforms*, Proc. Amer. Math. Soc. **133** (2005), 321-330.
- [RV] F. Rodriguez-Villegas, *Hypergeometric families of Calabi-Yau manifolds. Calabi-Yau Varieties and Mirror Symmetry (Yui, Noriko (ed.) et al., Toronto, ON, 2001), 223-231, Fields Inst. Commun., 38, Amer. Math. Soc., Providence, RI, 2003.*
- [S1] Z.H. Sun, *Invariant sequences under binomial transformation*, Fibonacci Quart. **39** (2001), 324-333.
- [S2] Z.H. Sun, *Congruences concerning Legendre polynomials*, Proc. Amer. Math. Soc. **139** (2011), 1915-1929.
- [S3] Z.H. Sun, *Congruences involving $\binom{2k}{k}$ and $\binom{6k}{3k}$* , J. Number Theory, submitted.
- [S4] Z.H. Sun, *Congruences concerning Legendre polynomials II*, arXiv:1012.3898. <http://arxiv.org/abs/1012.3898>.
- [S5] Z.H. Sun, *On the quadratic character of quadratic units*, J. Number Theory **128** (2008), 1295-1335.
- [Su1] Z.W. Sun, *Open conjectures on congruences*, arXiv:0911.5665. <http://arxiv.org/abs/0911.5665>.
- [Su2] Z.W. Sun, *Super congruences and elliptic curves over \mathbb{F}_p* , arXiv:1011.6676. <http://arxiv.org/abs/1011.6676>.
- [Su3] Z.W. Sun, *On sums involving products of three binomial coefficients*, arXiv:1012.3141. <http://arxiv.org/abs/1012.3141>.
- [Su4] Z.W. Sun, *On sums of Apery polynomials and related congruences*, arXiv:1101.1946. <http://arxiv.org/abs/1101.1946>.
- [T] R. Tauraso, *Supercongruences for a truncated hypergeometric series*, arXiv:1007.3852. <http://arxiv.org/abs/1007.3852>.
- [vH] L. van Hamme, *Some conjectures concerning partial sums of generalized hypergeometric series*, in: p-adic Functional Analysis (Nijmegen, 1996), pp. 223-236, Lecture Notes in Pure and Appl. Math., Vol. 192, Dekker, 1997.